Unitarity Alternatives in the Reduced-action Model for Gravitational Collapse

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Abstract

Based on the ACV approach to transplanckian energies, the reduced-action model for the gravitational S-matrix predicts a critical impact parameter $b_c \sim R \equiv 2G\sqrt{s}$ such that S-matrix unitarity is satisfied in the perturbative region $b > b_c$, while it is exponentially suppressed with respect to s in the region $b < b_c$ that we think corresponds to gravitational collapse. Here we definitely confirm this statement by a detailed analysis of both the critical region $b \simeq b_c$ and of further possible contributions due to quantum transitions for $b < b_c$. We point out, however, that the subcritical unitarity suppression is basically due to the boundary condition which insures that the solutions of the model be ultraviolet-safe. As an alternative, relaxing such condition leads to solutions which carry short-distance singularities presumably regularized by the string. We suggest that through such solutions — depending on the detailed dynamics at the string scale — the lost probability may be recovered.

1 Introduction

Interest in the gravitational S-matrix at transplanckian energies [1–4] has revived in the past few-years [5–7], when explicit solutions of the so-called reduced-action model [4] have been found [5]. The model is a much simplified version of the ACV eikonal approach [1,3] to transplanckian scattering in string-gravity, and is valid in the regime in which the gravitational radius $R \equiv 2G\sqrt{s}$ is much larger than the string length $\lambda_s \equiv \sqrt{\alpha' \hbar}$, so that string-effects are supposed to be small.

The reduced-action model (sec. 2) was derived by justifying the eikonal form of the S-matrix at impact parameter b on the basis of string dynamics and by then calculating the eikonal itself (of order $\sim \frac{Gs}{\hbar} \gg 1$) in the form of a 2-dimensional action, whose power series in $\frac{R^2}{b^2}$ corresponds to an infinite sum of proper irreducible diagrams (the "multi-H" diagrams [3, 4]), evaluated in the high-energy limit. The model admits a quantum generalization [6] of the S-matrix in the form of a path-integral — with definite boundary conditions — of the reduced-action exponential itself.

The main feature of the model and of its boundary conditions is the existence of a critical impact parameter $b_c \sim R$ such that, for $b > b_c$ the S-matrix matches the perturbative series and is unitary, while for $b < b_c$ the field solutions are complex-valued and the elastic S-matrix is suppressed exponentially. The suppression exponent is of order $\frac{Gs}{\hbar} \sim \frac{R^2}{\lambda_P^2}$ (λ_P being the Planck length) or, if we wish, of the same order as the entropy of a black-hole of radius R. From various arguments we believe that in the region in which $b < b_c$ (that is, b is smaller than the gravitational radius), a classical gravitational collapse is taking place.

The model, in its simplest axisymmetric form, is formulated in terms of only one effective field $\rho(r^2)$ depending on the transverse radius squared $r^2 \equiv \boldsymbol{x}^2$ and is defined by $\rho(r^2) \equiv r^2 \left(1 - (2\pi R)^2 \frac{\mathrm{d}}{\mathrm{d}r^2}\phi\right)$, where $h = \nabla^2\phi$ determines the transverse gravitational field and the corresponding metric, which is of shock-wave type. A key role in the derivation of the above features is played by the boundary condition $\rho(0) = 0$ which avoids a possible singularity of h at $r^2 = 0$ and is the main cause of the suppression of the S-matrix for $b < b_c$. In fact, the complex solutions of the semiclassical approach for $b < b_c$ were interpreted at quantum level [6] as due to a tunnel effect, required in order to reach $\rho = 0$ at $r^2 = 0$, across a potential barrier occurring in the lagrangian.

At this point the question was (and is): do we reach S-matrix unitarity for $b < b_c$ by summing over inelastic processes? do we recover full information [8] from the scattering experiment in the collapse region? It was already found in [7], by semiclassical methods, that this is not the case, and that the unitarity defect persists when all inelastic channels are included, although in a way dependent on the rapidity phase space parameter Y. This result is puzzling because one would like to know where does the probability go if the model is complete or — if it is not — how to complete it.

The purpose of the present paper is both to look at possible flaws in the result just quoted and to suggest a tentative answer to the ensuing question. We exclude flaws in two ways. In sec. 3 we perform a detailed analysis of the solutions of the semiclassical unitarity equations, we choose the stable ones and we investigate the unitarity behaviour by a perturbative method around the critical point $b \simeq b_c$ and by numerical methods elsewhere. There are no surprises: the results of [7] are fully checked, we only gain some better understanding of the unitarity defect around the critical point.

As a second attempt, we look for a quantum treatment of inelastic S-matrix elements (sec. 4). Since the quantum version of the reduced action model features a quantum mechanical hamiltonian with a Coulomb potential in ρ-space, we eventually evaluate quantum transitions from the basic tunneling wave function to other states of the system. Of particular interest are the bound states in the strong-coupling region $\rho \leq 0$ where the potential is attractive, because they could correspond to collapsed matter. Unfortunately, all relevant matrix elements carry the same exponential suppression as the tunneling amplitude itself, and the unitarity defect survives.

As a final point, in sec. 5 we test the boundary conditions of the model, by letting $\rho(0)$ fluctuate away from zero, with the weight assigned to it by the reduced action itself. We find that, while the elastic S-matrix element is stable — that is, dominated by very small $\rho(0)$ — large inelastic contributions may come from the short-distance region, where however the model is inadequate and string effects are expected to play an important role. We argue on this basis that in the direction of such ultraviolet-sensitive solutions the model is incomplete and that, by completing it with the proper string dynamics we may discover where the probability goes.

2 The reduced-action model for gravitational $\boldsymbol{S}$-matrix

We provide here a brief account of the model being considered — as develope in refs. [5,6] — for both completeness sake and in order to emphasize some points which are useful in the following.

2.1 The semiclassical ACV results

The simplified ACV approach [5] to transplanckian scattering in the regime $R \equiv 2G\sqrt{s} \gg \lambda_s$ is based on two main points. Firstly, the gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ associated to the high-energy scattering of light particles, reduces to a shock-wave configuration of the form

$$h_{--}\big|_{x^+=0} = (2\pi R) a(\boldsymbol{x}) \delta(x^-) \; , \qquad h_{++}\big|_{x^-=0} = (2\pi R) \bar{a}(\boldsymbol{x}) \delta(x^+) \tag{1a}$$

$$h_{ij} = (\pi R)^2 \Theta(x^+ x^-) \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) h(\boldsymbol{x}) \; , \tag{1b}$$

where a, $\bar{a}$ are longitudinal profile functions, and $h(\boldsymbol{x}) \equiv \nabla^2 \phi$ is a scalar field describing one emitted-graviton polarization (the other, related to soft graviton radiation, is negligible in an axisymmetric configuration).

Secondly, the high-energy dynamics itself is summarized in the h-field emission-current $\mathcal{H}(\boldsymbol{x})$ generated by the external sources coupled to the longitudinal fields a and $\bar{a}$. Such a vertex has been calculated long ago [9, 10] and takes the form

$$-\nabla^2 \mathcal{H} \equiv \nabla^2 a \nabla^2 \bar{a} - \nabla_i \nabla_j a \nabla_i \nabla_j \bar{a} \; , \tag{2}$$

which is the basis for the gravitational effective action [11–13] from which the shock-wave solution (1) emerges [4]. It is directly coupled to the field h and, indirectly, to the external sources s and $\bar{s}$ in the reduced 2-dimensional action

$$\frac{\mathcal{A}}{2\pi G s} = \int \mathrm{d}^2\boldsymbol{x} \left[a\bar{s} + \bar{a}s - \frac{1}{2} \nabla a \nabla \bar{a} + \frac{(\pi R)^2}{2} \left(-(\nabla^2 \phi)^2 - 2 \nabla \phi \cdot \nabla \mathcal{H} \right) \right] \tag{3}$$

which is the basic ingredient of the ACV simplified treatment. Note that here the gravitational radius R plays the role of (dimensionful) coupling constant and that — because of the higher derivatives of ϕ involved — non-renormalizable UV divergences may occur in general.

The equations of motion (EOM) induced by (3) provide, with proper boundary conditions, some well-defined effective metric fields a and h which are, hopefully, UV-safe. The "on-shell" action $\mathcal{A}(b, s)$, evaluated on such fields, provides directly the elastic S-matrix

$$S_{\mathrm{el}} = \exp\left(\frac{\mathrm{i}}{\hbar} \mathcal{A}(b, s) \right) . \tag{4}$$

Then, it can be shown [4, 5] that the reduced-action above (where now R/b plays the role of effective coupling constant) resums the so-called multi-H diagrams (fig. 1), contributing a series of corrections $\sim (R^2/b^2)^n$ to the leading eikonal, as well as their resummation for $R/b = \mathcal{O}(1)$.

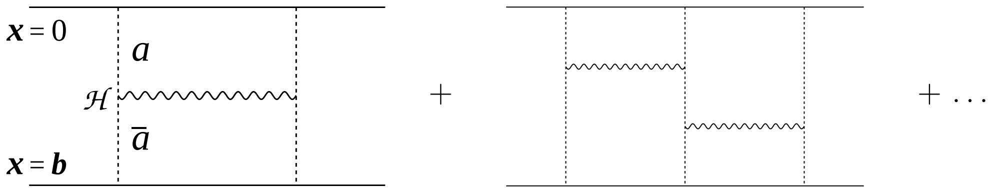


Figure 1: *Diagrammatic series of H and multi-H diagrams.*

Furthermore, the S-matrix (4) can be extended to inelastic processes on the basis of the same emitted-graviton field $h(\boldsymbol{x})$. In the eikonal formulation the inelastic S-matrix is approximately[1] described by the coherent state operator

$$S = \exp\left(\frac{\mathrm{i}}{\hbar}\mathcal{A}(b,s)\right)\exp\left(\mathrm{i}2\pi R\sqrt{\alpha}\int \mathrm{d}^2\boldsymbol{x}\; h(\boldsymbol{x})\Omega(\boldsymbol{x})\right)\,, \qquad \alpha \equiv \frac{Gs}{\hbar} \tag{5}$$

$$\Omega(\boldsymbol{x}) \equiv \int \frac{\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}k_3}{2\pi\sqrt{k_0}}\left[a(\boldsymbol{k},k_3)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + h.c.\right] \equiv A(\boldsymbol{x}) + A^\dagger(\boldsymbol{x})\,, \tag{6}$$

$$[A(\boldsymbol{x}), A^\dagger(\boldsymbol{x}')] = Y\delta(\boldsymbol{x}-\boldsymbol{x}')$$

where the operator $\Omega(\boldsymbol{x})$ incorporates both emission and absorption of the h-fields and Y parameterizes the rapidity phase space which is effectively allowed for the production of light particles (e.g. gravitons).

For a given value of the "gravitational coupling" $\alpha \equiv Gs/\hbar$ the parameter Y is possibly large for large impact parameters $b \gg \sqrt{G\hbar}$, because the effective transverse mass of the light particles is expected to be of order $\hbar/b$, i.e., much smaller than the Planck mass, thus yielding roughly $Y \sim \log(sb^2/\hbar^2) \gg 1$. On the other hand, we should notice that dynamical arguments based on energy conservation [14] and on absorptive corrections of eikonal type, consistent with the AGK cutting rules [15], tend to suppress the fragmentation region in a b-dependent way, so as to constrain Y to be $\mathcal{O}(1)$ for impact parameters in the classical collapse region $b = \mathcal{O}(R)$. However, such arguments do not take into account possible dynamical correlations coming from multi-H diagrams, as mentioned in footnote 1. It is fair to state that a full dynamical understanding of the Y parameter is not available yet, and for this reason we shall discuss what happens for any values of Y.

In the case of axisymmetric solutions, where $a = a(r^2)$, $\bar{a} = \bar{a}(r^2)$, $\phi = \phi(r^2)$ it is straightforward to see, by using eq. (2), that $\dot{\mathcal{H}}(r^2) \equiv (\mathrm{d}/\mathrm{d}r^2)\mathcal{H}(r^2) = -2\dot{a}\dot{\bar{a}}$ becomes proportional to the $a, \bar{a}$ kinetic term. Therefore, the action (3) can be rewritten in the more compact one-dimensional form

$$\frac{\mathcal{A}}{2\pi^2 Gs} = \int \mathrm{d}r^2\left[a(r^2)\bar{s}(r^2) + \bar{a}(r^2)s(r^2) - 2\rho\dot{\bar{a}}\dot{a} - \frac{2}{(2\pi R)^2}(1-\dot{\rho})^2\right]\,, \qquad \dot{a} \equiv \frac{\mathrm{d}a}{\mathrm{d}r^2}\,, \tag{7}$$

where we have introduced the auxiliary field $\rho(r^2)$

$$\rho = r^2\big(1-(2\pi R)^2\dot{\phi}\big)\,, \qquad h = 4\dot{(r^2\dot{\phi})} = \frac{1}{(\pi R)^2}(1-\dot{\rho}) \tag{8}$$

[1]The coherent state describes uncorrelated emission (apart from momentum conservation [14]). However, the eikonal approach based on eq. (3) also predicts [5] correlated particle emission, which is suppressed by a power of $(Gs/\hbar)Y$ relative to the uncorrelated one, and is not considered here.

which incorporates the ϕ-dependent interaction. The external sources $s(r^2)$, $\bar{s}(r^2)$ are assumed to be axisymmetric also, and are able to approximately describe the particle-particle case by setting $\pi s(r^2) = \delta(r^2)$, $\pi\bar{s}(r^2) = \delta(r^2 - b^2)$, where the azimuthal averaging procedure of ACV is assumed.[2]

The equations of motion, specialized to the case of particles at impact parameter b have the form

$$\dot{a} = -\frac{1}{2\pi\rho}\,, \qquad \dot{\bar{a}} = -\frac{1}{2\pi\rho}\Theta(r^2 - b^2)\,, \tag{9}$$

$$\ddot{\rho} = \frac{1}{2\rho^2}\Theta(r^2 - b^2) \qquad (r > b) \tag{10}$$

and show a "Coulomb" potential in ρ-space, which is repulsive for $\rho > 0$, acts for $r > b$ and plays an important role in the tunneling phenomenon. By replacing the EOM (9) into eq. (7), the reduced action can be expressed in terms of the ρ field only, and takes the simple form

$$\mathcal{A} = -Gs \int \mathrm{d}r^2 \left(\frac{1}{R^2}(1-\dot{\rho})^2 - \frac{1}{\rho}\Theta(r^2 - b^2) \right) \equiv -\int_0^\infty \mathrm{d}r^2\; L(\rho, \dot{\rho}, r^2)\,, \tag{11}$$

which is the one we shall consider at quantum level in the following.

The effective metric (1) generated by the axisymmetric fields ρ, a and $\bar{a}$ is calculated [5] on the basis of the complete form of the shock-wave (1) and is given by

$$\begin{aligned}
\mathrm{d}s^2 = &-\mathrm{d}x^+\mathrm{d}x^- \left[1 - \tfrac{1}{2}\Theta(x^+x^-)(1-\dot{\rho})\right] \\
&+ (\mathrm{d}x^+)^2\delta(x^+)\left[2\pi R\bar{a}(r^2) - \tfrac{1}{4}(1-\dot{\rho})|x^-|\right] \\
&+ (\mathrm{d}x^-)^2\delta(x^-)\left[2\pi R a(r^2) - \tfrac{1}{4}(1-\dot{\rho})|x^+|\right] \\
&+ \mathrm{d}r^2\left[1 + 2(\pi R)^2\Theta(x^+x^-)\dot{\phi}\right] + \mathrm{d}\theta^2\, r^2\left[1 + 2(\pi R)^2(\dot{\phi} + r^2\ddot{\phi})\right]
\end{aligned} \tag{12}$$

This metric is dynamically generated and may be regular or singular at short distances, depending on the behaviour of the field solutions themselves.

2.2 Boundary conditions and critical impact parameter

Two boundary conditions are necessary to solve the equation of motion (10): first of all we set $\dot{\rho}(\infty) = 1$ in order to have a gravitational field $h \sim 1 - \dot{\rho}$ vanishing at large distances. The second boundary condition is $\rho(0) = 0$ and it is necessary in order to obtain UV-safe solutions at short distances. Indeed, by the definition of ρ — which embodies an r^2 factor in eq. (8) —, the fields ϕ and h are singular if $\rho(0) \neq 0$. More precisely $\dot{\phi} \simeq -\rho(0)/r^2$ is quadratically divergent, and this implies that the outgoing flux of $\nabla\phi$ at the origin is $\simeq -\rho(0)$ and therefore $h \equiv \nabla^2\phi \simeq -\rho(0)\delta(r^2)$ is singular too. Thus, solutions with $\rho(0) \neq 0$ possess UV singularities, and this implies in turn an UV divergent action and a singular metric. In particular, the singularity of $\dot{\phi}$ is present in the metric coefficient h_{rr}, which changes sign also, while a $\delta(r^2)$ singularity of difficult interpretation occurs in the

[2]The most direct interpretation of this configuration is the scattering of a particle off a ring-shaped null matter distribution, which is approximately equivalent to the particle-particle case by azimuthal averaging [5].

rest of the metric (12). For these reasons we set $\rho(0) = 0$ so as to obtain a finite action and a regular metric.

The solution of the equation of motion satisfying the above boundary conditions is

$$\rho = \begin{cases} t_b r^2 & (r^2 < b^2) \\ R^2 \cosh^2 \chi(r^2) & (r^2 \geq b^2) \\ r^2 - b^2 = R^2(\chi + \sinh\chi\cosh\chi - \chi_b - \sinh\chi_b \cosh\chi_b) \end{cases} \tag{13}$$

with $\tanh \chi_b = t_b = \dot{\rho}(b^2)$ and $\chi_b \equiv \chi(b^2)$. The parameter t_b is determined by requiring regular matching of the solution at $r^2 = b^2$, that is

$$t_b(1 - t_b^2) = \frac{R^2}{b^2} \tag{14}$$

This equation acquires the meaning of criticality equation. In fact it can be solved[3] only if the impact parameter b exceeds a critical value b_c given by $b_c^2 = \frac{3\sqrt{3}}{2}R^2$. For $b > b_c$, one of the two real solutions matches the perturbative result [3, 5]

$$\mathcal{A}(b, s) = Gs\left(\log\frac{L^2}{b^2} + \frac{R^2}{2b^2} + \cdots\right) , \tag{15}$$

where L is an infrared cutoff parameterizing the infinite "Newtonian" phase. If instead $b < b_c$, the criticality equation has two complex conjugate solutions, providing an imaginary part of ρ in eq. (13) that we can't interpret at purely classical level. Anyway we can try to use these complex solutions and substitute them in the on shell action, which is easily calculated to be

$$\mathcal{A} = Gs\left\{2[\chi(L^2) - \chi_b] - \frac{1 - t_b}{t_b}\right\} \tag{16}$$

where $\chi(L^2) \simeq \log(L/R)$.

For $b < b_c$ this expression becomes complex-valued, providing a non unitary $S = \mathrm{e}^{\mathrm{i}\mathcal{A}}$. It can be shown by stability arguments [5] that the physical solution of (14), when $b < b_c$, is the one with negative imaginary part of t_b, giving rise to a suppression of the S-matrix in the elastic channel. For example, for $b = 0$, we obtain $\chi_{b=0} = -\mathrm{i}\pi/2$ and the exponential suppression

$$S \sim \mathrm{e}^{-\pi Gs} . \tag{17}$$

The suppression exponent $Gs \sim R^2/\lambda_P^2$ is of the same order as the entropy of a black-hole of radius R, but occurs here because of the complex-valued $\rho(r^2)$ when the impact parameter is smaller than the critical value.

We conclude this subsection by noting that real-valued solutions to eq. (10) and satisfying $\dot{\rho}(\infty) = 1$ are always of the form (13) and have necessarily $\rho(0) > 0$ whenever $b < b_c$. The minimum (and closest to 0) value of $\rho(0)$ is found for a particular value of the initial slope $\dot{\rho}(0) = t_m$ determined by

$$\frac{b^2}{2R^2} = \cosh^3(\chi_m)\sinh(\chi_m) = \frac{t_m}{(1 - t_m^2)^2} . \tag{18}$$

The corresponding solution provides, so to say, the real solution with smallest "distance" to the complex solution having $\rho(0) = 0$.

[3] We consider only solutions with $t_b > 0$, otherwise the condition $\dot{\rho}(\infty) = 1$ would not be satisfied.

2.3 The quantized S-matrix and tunnel effect

The suppression of the S-matrix for $b < b_c$ can be interpreted by generalizing the scattering matrix to a quantum level. In the CC proposal [6] this is achieved by summing the reduced-action exponential over every path $\rho(r^2)$ satisfying the boundary conditions $\rho(0) = 0$ and $\dot\rho(\infty) = 1$

$$S(b,s) = \int_{\rho(0)=0}^{\dot\rho(\infty)=1} [\mathcal{D}\rho(r^2)]\ \mathrm{e}^{-\mathrm{i}\int L(\rho,\dot\rho,r^2)\mathrm{d}r^2}\ \mathrm{e}^{\frac{2\mathrm{i}\sqrt{\alpha}}{\pi R}\int[1-\dot\rho]\Omega(\boldsymbol{x})\ \mathrm{d}^2\boldsymbol{x}}\ , \tag{19}$$

where we have included the coherent state operator Ω of eq. (6) in order to describe inelastic processes. The S-matrix in the elastic channel is obtained by evaluating the vacuum expectation value and we obtain

$$S_{\mathrm{el}} = \int_{\rho(0)=0}^{\dot\rho(\infty)=1} [\mathcal{D}\rho]\ \mathrm{e}^{-\mathrm{i}\int L_y(r^2)\ \mathrm{d}r^2} \tag{20}$$

$$L_y = \frac{1}{4G}\left[(1-\mathrm{i}y)(1-\dot\rho)^2 - \frac{R^2}{\rho}\Theta(r^2-b^2)\right]\ , \qquad y \equiv \frac{2Y}{\pi}\ . \tag{21}$$

The parameter y is related to the rapidity phase space allowed for the emitted gravitons, and provides a related absorption of the elastic channel. In this section, we will examine only the case $y = 0$ and $L_y \to L$, neglecting such absorptive effects. In this way, the model is less complicated, nevertheless it explains the suppression (17). The general case $y \neq 0$ has been discussed in [7].

The problem of calculating S_{el}, by use of the Trotter formula, turns out to be equivalent to quantize a hamiltonian and to evaluate a matrix element of the evolution operator $\mathcal{U}(0,\infty)$. In other words, the classical dynamics of the field $\rho(r^2)$ governed by the lagrangian (11), is promoted to a one-dimensional quantum system where ρ plays the role of "position variable" and $r^2 \equiv \tau$ represents the "time" evolution variable. The classical hamiltonian is given by the Legendre transform of L and we quantize it by imposing canonical commutation relations: introducing the conjugate momentum $\Pi = \frac{\partial L}{\partial\dot\rho} = \frac{1}{2G}(\dot\rho - 1)$, we have

$$H = \Pi\dot\rho - L = \frac{1}{4G}\left(\dot\rho^2 - 1 + \frac{R^2}{\rho}\Theta(\tau - b^2)\right)\ , \qquad \tau \equiv r^2 \tag{22}$$

$$[\rho, \Pi] = \mathrm{i}\hbar \quad \longrightarrow \quad \dot\rho = -\mathrm{i}\frac{R^2}{2\alpha}\frac{\partial}{\partial\rho}\ . \tag{23}$$

The Hamiltonian, according to this quantization, is

$$\frac{\hat H}{\hbar} = -\frac{R^2}{4\alpha}\frac{\partial^2}{\partial\rho^2} + \alpha\left(\frac{\Theta(\tau-b^2)}{\rho} - \frac{1}{R^2}\right) \equiv \frac{H_0}{\hbar} + \alpha\frac{\Theta(\tau-b^2)}{\rho} \tag{24}$$

In this way, the path integral (20) is equivalent to the matrix element

$$S(b,s) = \langle\rho = 0|\mathcal{U}(0,+\infty)|\Pi = 0\rangle \tag{25}$$

where the initial and final states express the boundary conditions for ρ (we recall the relation $|\Pi = 0\rangle = |\dot\rho = 1\rangle$). We note that the Hamiltonian is characterized by a Coulomb

potential that is attractive for $\rho < 0$ and repulsive in the region $\rho > 0$: there is an infinite Coulomb barrier separating the boundary condition $\rho = 0$ from the (perturbative) region of large $\rho > 0$ and by means of this feature we'll be able to interpret the suppression of the S-matrix as a tunnel effect.

The quantum model can be solved for any b, but is particularly simple for $b = 0$, where it explains the value $S \sim \mathrm{e}^{-\pi G s}$ obtained at semiclassical level. Indeed, if the impact parameter vanishes, the expression (25) gives

$$S(b = 0, s) = \langle \rho = 0 | \mathcal{U}_c(0, +\infty) | \Pi = 0 \rangle \tag{26}$$

where $\mathcal{U}_c$ is the evolution operator related to the time independent Coulomb hamiltonian $H_c = H_0 + Gs/\rho$. We calculate the above matrix element by introducing the Coulomb wave function

$$\psi_c(\rho) = \langle \rho | \mathcal{U}_c(0, \infty) | \Pi = 0 \rangle \tag{27}$$

so that $S(b = 0, s) = \psi_c(0)$. Therefore, the $b = 0$ quantum problem is very similar to the calculation of the Coulomb wave function at the origin for nuclear processes. It can be shown [7] that, for every b, the S-matrix is given by the formula

$$S(b, s) = \sqrt{\frac{\mathrm{i}\alpha}{\pi b^2}} \int \mathrm{d}\rho \; \mathrm{e}^{-\mathrm{i}\alpha(\frac{\rho^2}{b^2} + b^2)} \, \psi_c(\rho) \tag{28}$$

which was treated in full details in refs. [6, 7].

In order to derive the function $\psi_c(\rho)$, we note that the state $|\Pi = 0\rangle$ is an eigenstate of the free hamiltonian H_0 with zero energy, so that $|\psi_c\rangle$ becomes an eigenstate of the full hamiltonian with null eigenvalue. Therefore, the wave function can be determined by solving the stationary Schrodinger equation with zero energy (from now on we use $R = 1$ as unit length)

$$H_c \, \psi_c = \hbar \left[-\frac{1}{4\alpha} \frac{\mathrm{d}^2 \psi_c}{\mathrm{d}\rho^2} + \alpha \left(\frac{1}{\rho} - 1 \right) \psi_c \right] = 0 \; . \tag{29}$$

The form of ψ_c is specified, including its boundary conditions, by the Lippman-Schwinger equation

$$\psi_c = \mathrm{e}^{2\mathrm{i}\alpha\rho} + \alpha G_0(0) \, \mathrm{pv} \left(\frac{1}{\rho} \right) \psi_c(\rho) \tag{30}$$

with $G(E) = [E - H_0 - \mathrm{i}\epsilon]^{-1}$. [4] For $\rho > 0$, where the potential is repulsive, the Coulomb function contains incident and reflected waves, while it has only a transmitted wave in the region $\rho < 0$ of attractive potential. This explains why the wave function $\psi_c(\rho)$ is suppressed in the origin by the tunnel effect through the Coulomb barrier and the tunneling amplitude gives the order of magnitude of the S-matrix for $b = 0$.

Indeed, by solving eq. (29) with the above boundary conditions, the wave function ψ_c turns out to be [7]

$$\psi_c = \frac{(4\mathrm{i}\alpha L^2)^{i\alpha}}{\cosh(\pi\alpha)} z \mathrm{e}^{-\frac{z}{2}} \left[U(1 + \mathrm{i}\alpha, 2, z) + \frac{\mathrm{i}\pi\Theta(\mathrm{i}z)}{\Gamma(\mathrm{i}\alpha)} F(1 + \mathrm{i}\alpha, 2, z) \right] \; , \qquad (z \equiv -4\mathrm{i}\alpha\rho) \tag{31}$$

[4] The $-\mathrm{i}\epsilon$ prescription is related to the r^2-antiordering of $\mathcal{U}_c(0, \infty)$.

where L is an infrared cutoff that regularizes the Coulomb singularity at large distances while U and F are the irregular and regular confluent hypergeometric functions respectively. The value at the origin of the wave function is

$$\psi_c(0) = S(0,s) = \frac{(4\alpha L^2)^{\mathrm{i}\alpha}\mathrm{e}^{-\frac{\pi\alpha}{2}}}{\Gamma(1+\mathrm{i}\alpha)\cosh(\pi\alpha)} \sim \mathrm{e}^{-\pi G s}\,\mathrm{e}^{\mathrm{i}\Re\mathcal{A}_{\mathrm{cl}}}\left[1+\mathcal{O}\left(\frac{1}{\alpha}\right)\right] . \tag{32}$$

In this way we obtain an interpretation for the suppression of the elastic amplitude for $b < b_c$, as well as quantum corrections to it.

3 Semiclassical unitarity defect below the critical point

The reduced-action model described in the previous section shows the existence of a critical impact parameter $b_c \sim R(s)$ such that, for $b < b_c$ the elastic scattering amplitude is exponentially suppressed with s both at semiclassical and quantum level, even without inelastic processes ($y = 0$). For $b > b_c$ the small elastic absorption which appears in the quantum model at $y > 0$ is expected to be compensated by emission processes. We do not know about $b < b_c$, but it might be possible that the elastic suppression due to the tunneling is still compensated, because of the quantum gravity dynamics.

In any case, the model can be unitary only if we consider the total transition probability, summed over all emission processes. For instance, at $b = 0$, the suppression factor

$$|\langle 0|S|0\rangle|^2 \sim \mathrm{e}^{-2\pi\alpha} \tag{33}$$

could be compensated, thus recovering unitarity, only in the sum of all probabilities

$$\sum_n |\langle n|S|0\rangle|^2 = \langle 0|S^\dagger S|0\rangle \stackrel{?}{=} 1 , \tag{34}$$

where $|n\rangle$ generically indicates "the" state with n gravitons which can be emitted thanks to the coherent state operator in eqs. (6,19).

3.1 Inclusive action and equations of motion

In order to study the unitarity properties of our model, we try to evaluate the vacuum expectation value of $S^\dagger S$. From the definition (19), it is clear that the result will be a double path-integral in the fields $\rho(\tau)$ [from S] and $\tilde\rho(\tau)$ [from $S^\dagger$]. The matrix element between the vacuum states is computed after normal ordering of the operators $A, A^\dagger$ in the coherent-state operators by means of the Baker-Campbell-Hausdorff formula

$$\langle 0|\mathrm{e}^{-\frac{2\mathrm{i}\sqrt{\alpha}}{\pi R}\int \mathrm{d}^2\boldsymbol{x}'\,[1-\dot{\tilde\rho}(\tau')]\Omega(\boldsymbol{x}')}\mathrm{e}^{\frac{2\mathrm{i}\sqrt{\alpha}}{\pi R}\int \mathrm{d}^2\boldsymbol{x}\,[1-\dot{\rho}(\tau)]\Omega(\boldsymbol{x})}|0\rangle = \mathrm{e}^{\alpha y\int[\dot{\tilde\rho}(\tau)-\dot\rho(\tau)]^2\,\mathrm{d}\tau} . \tag{35}$$

We obtain the expression

$$\langle 0|S^\dagger S|0\rangle = \int[\mathcal{D}\rho][\mathcal{D}\tilde\rho]\;\mathrm{e}^{\mathrm{i}\mathcal{A}_u(\rho,\tilde\rho;b)} , \tag{36}$$

where the usual boundary conditions $\rho(0) = \tilde\rho(0) = 0$, $\dot\rho(\infty) = \dot{\tilde\rho}(\infty) = 1$ are understood and we introduced the *inclusive action*

$$\mathcal{A}_u \equiv -\alpha\int_0^\infty\left[(1-\dot\rho)^2 - \frac{\Theta(\tau-b^2)}{\rho} - (1-\dot{\tilde\rho})^2 + \frac{\Theta(\tau-b^2)}{\tilde\rho} - \mathrm{i}y(\dot\rho-\dot{\tilde\rho})^2\right]\mathrm{d}\tau . \tag{37}$$

Here the y-dependent term shows the effect due to summing over all intermediate states.

In the spirit of the semiclassical approximation, a good estimate of the value of the functional integral for large α, i.e., for transplanckian energies, is obtained by evaluating the inclusive action on the field configuration that maximizes $\mathrm{i}\mathcal{A}_u$. The inclusive action is stationary when the fields ρ and $\tilde{\rho}$ satisfy the Euler-Lagrange equations

$$\begin{cases} 2\ddot{\rho} - 2\mathrm{i}y(\ddot{\rho} - \ddot{\tilde{\rho}}) & = \dfrac{\Theta(\tau - b^2)}{\rho^2} \\ 2\ddot{\tilde{\rho}} + 2\mathrm{i}y(\ddot{\tilde{\rho}} - \ddot{\rho}) & = \dfrac{\Theta(\tau - b^2)}{\tilde{\rho}^2} \end{cases} \tag{38}$$

and the condition of maximum for $\mathrm{i}\mathcal{A}_u$ must be determined independently for the various solutions by studying the stability of the action functional for arbitrary variations of the fields (cfr. sec. 3.2).

At the semiclassical level we are considering, it is assumed that the sum over field configurations represented by the path integral be dominated by just one solution among those satisfying the Euler-Lagrange equations (38). Therefore, if ρ is such a solution yielding the dominant contribution to S, for symmetry reasons the solution $\tilde{\rho}$ yielding the dominant contribution to $S^\dagger$ must be equal to ρ^*. In other words, we argue that only solutions with $\rho(\tau) = \tilde{\rho}^*(\tau) \equiv \rho_1 + \mathrm{i}\rho_2$ are physically acceptable within the semiclassical approximation, $\rho_{1,2}$ being real fields.

Once the semiclassical solutions $\rho = \rho_1 + \mathrm{i}\rho_2$ have been determined, the action is calculated by splitting the integration over τ in eq. (37) into two pieces: *(i)* the first interval $0 \leq \tau \leq b^2$ where the fields freely evolve and the integrand is constant; *(ii)* the second interval $b^2 < \tau < \infty$ where the evolution is non-trivial but admits a constant of motion provided by the hamiltonian obtained by Legendre transform of the lagrangian in eq. (37):

$$H_u = 2\dot{\rho}_1\dot{\rho}_2 + 2y\dot{\rho}_2^2 - \frac{\rho_2}{\rho_1^2 + \rho_2^2} = 0 \, . \tag{39}$$

No additional constant of motion exists, so that the system (38) is not solvable analytically. Nevertheless, various relations [7] allow one to express the inclusive action in terms of few parameters of the solution, namely the slope $t = t_1 + \mathrm{i}t_2$ of $\rho(\tau)$ in the free region *(i)*, and the asymptotic value $\rho_\infty \equiv \rho_2(\infty)$, according to the formula [7]

$$\mathrm{i}\mathcal{A}_u = 4\alpha \left[\rho_\infty - \frac{3}{2}\frac{t_2}{t_1^2 + t_2^2} \right] \, . \tag{40}$$

This equation shows that the inclusive action vanishes for real solutions ($\rho_2 = 0$) in which case the S-matrix is unitary. On the contrary, absorption is present only with a non-vanishing imaginary part ρ_2. By studying the "inclusive" equation of motion (38), we can determine for which values of b and y the solutions necessarily develop an imaginary part, and we can then compute the unitarity defect. However, since no analytic solution is available, we cannot exhibit an explicit criticality equation, and we have to resort to numerical or approximate methods.

3.2 Real and complex inclusive solutions

In the previous section we showed that a sufficient condition for unitarity is the existence of real solutions $\rho = \rho^* = \tilde{\rho}$, because the inclusive action vanishes identically, thus implying

$\langle 0|S^\dagger S|0\rangle = 1$ at semiclassical level. In this case both equations in (38) reduce to the equation (10) governing the elastic S-matrix, with the same boundary conditions, hence with real solutions only for $b > b_c$. As a consequence, in this regime the model is unitary. This argument shows that the same critical point b_c — found to be $b_c^2 = \frac{3\sqrt{3}}{2} R^2$ for the elastic channel at $y = 0$ — also governs the inelastic unitarity of the S-matrix in a y-independent way.

The above reasoning also implies that, below the critical impact parameter ($b < b_c$), only genuine complex solutions of (38) exist. In order to characterize such solutions, we have written a numerical program that seeks for independent pairs $(\rho, \tilde{\rho})$ satisfying the inclusive system. We found that, without imposing the "physical" condition $\tilde{\rho} = \rho^*$, there are four distinct complex solutions for any given value of b and y. By labelling each solution with the value of $t \equiv \dot{\rho}(0)$, we can represent the four solutions with four points in the complex t-plane. At fixed y and increasing b these points move, spanning the curves $M_1 \cup M_3$, $M_2 \cup M_4$, $N_1 \cup N_3$ and $N_2 \cup N_4$ depicted in fig. 2a in the direction of the arrows. Now, if we consider only the solutions with $\rho = \tilde{\rho}^*$, their number varies with b according to the following scheme:

- $b < b_c$: there are two complex solutions, one with positive (N_1) and one with negative (N_2) imaginary part;
- $b_c < b < b_d(y)$: there are two real (M_3, M_4) and two complex (N_1, N_2) solutions, the latter with positive imaginary part; here b_d is a y-dependent critical value that will be analytically described in the next section; [5]
- $b > b_d(y)$: only the real solutions (M_3, M_4) survive.

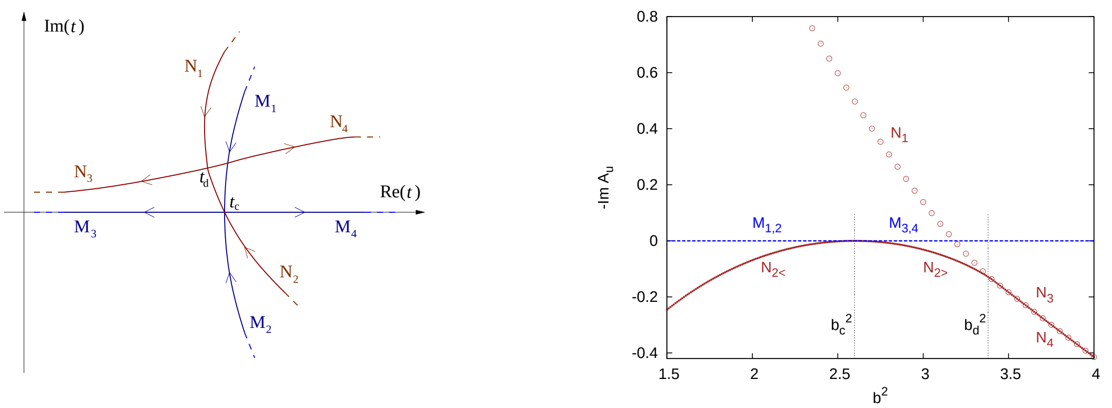


Figure 2: [Left] *Regions of t-values spanned by varying b at fixed $y = 0.5$.* [Right] *The inclusive action corresponding to the various solutions.*

As already mentioned, we can accept only field configurations corresponding to maxima of i$\mathcal{A}_u$. Analytically, stability is expressed by requiring the second variation of $\Im \mathcal{A}_u$ to be positive definite with respect to arbitrary *real* variations $(\delta\rho, \delta\tilde{\rho})$ of the solution.[6]

[5]At $b = b_c$ the three solutions M_1, M_2 and N_2 are pure real and coincide. At $b = b_d$ the two solutions N_1, and N_2 coincide.

[6]We consider only real variations of ρ and $\tilde{\rho}$ for compatibility with the definition of the path integral (19) which is defined by integration over real fields.

By definition, the first-order variation of the inclusive action vanishes on the solutions: $\delta\mathcal{A}_u = 0$. The imaginary part of the second-order variation yields

$$\delta^2 \Im \mathcal{A}_u = \int \left\{ y(\delta\dot{\rho} - \delta\dot{\tilde{\rho}})^2 - \Theta(\tau - b^2) \frac{\rho_2(3\rho_1^2 - \rho_2^2)}{(\rho_1^2 + \rho_2^2)^3} \left[(\delta\rho)^2 + (\delta\tilde{\rho})^2\right] \right\} \mathrm{d}\tau \;, \tag{41}$$

which is positive definite provided

$$\rho_2(3\rho_1^2 - \rho_2^2) < 0 \;. \tag{42}$$

In practice, when $|\rho_2|$ is smaller than $|\rho_1|$, as happens in the important region $b \sim b_c$, stability occurs for $\rho_2 \leq 0$. Therefore, the only complex solution satisfying conditions (42) is found on $N_2|_{\Im t<0}$ for $b < b_c$; all other complex solutions are unstable. On the other hand, any real solution is stable.

To summarize, for $b > b_c$ we have two real and y-independent acceptable solutions which coincide with the exclusive solutions studied in [6], providing a unitary S-matrix. For $b < b_c$ instead, we have only one complex acceptable solution with negative imaginary part ($\Im(t) < 0$) and corresponding negative action $\mathrm{i}\mathcal{A}_u < 0$ (cfr. fig. 2b), thus determining a unitarity defect for the S-matrix.

3.3 Behaviour around the critical point

In order to investigate the onset of absorption below the critical point and to quantify its magnitude, we study the behaviour of the solutions of the inclusive equations (38) for $b \simeq b_c$. In particular, we consider the physical ($\rho = \tilde{\rho}^*$) complex solution N_2 which provides the unitarity defect for $b < b_c$. The fact that this solution becomes real ($\rho_2(\tau) = 0$) for $b = b_c$, suggests to perform a perturbative analysis in which the imaginary part ρ_2 is considered a small quantity in some neighbourhood of $b \simeq b_c$.

We first set up the relevant equations in a way that is convenient for the perturbative expansion. In terms of the real components $\rho_{1,2}$, eqs. (38) read

$$\begin{cases} 2\ddot{\rho}_1 + 4y\ddot{\rho}_2 = \Theta(\tau - b^2) \dfrac{\rho_1^2 - \rho_2^2}{(\rho_1^2 + \rho_2^2)^2} & \text{(43a)} \\ 2\ddot{\rho}_2 = \Theta(\tau - b^2) \dfrac{-2\rho_1\rho_2}{(\rho_1^2 + \rho_2^2)^2} & \text{(43b)} \\ \rho_1(0) = 0 \;, \quad \rho_2(0) = 0 \;, \quad \dot{\rho}_1(\infty) = 1 \;, \quad \dot{\rho}_2(\infty) = 0 \;. & \text{(43c)} \end{cases}$$

The evolution for $\tau \leq b^2$ is linear in τ:

$$\rho_{1,2}(\tau) = t_{1,2}\,\tau \;, \qquad\qquad (\tau \leq b^2) \;. \tag{44}$$

For $\tau \geq b^2$, thanks to the existence of the integral of motion (39), we can reduce by one the order of the differential system (43), e.g. replacing the system (43) with the linear combination (43a)$-2y$(43b) and eq. (39) itself.

A convenient way of rewriting eq. (39) is to consider ρ_1 the independent variable, i.e., $\rho_2 = \rho_2\big(\rho_1(\tau)\big)$. This is possible since $\rho_1(\tau)$ turns out to be a monotonic increasing function of τ. By denoting with a prime the derivative with respect to ρ_1, we have

$$\rho_2' \equiv \frac{\mathrm{d}\rho_2}{\mathrm{d}\rho_1} = \frac{\dot{\rho}_2}{\dot{\rho}_1} \;. \tag{45}$$

Dividing eq. (39) by $(\dot{\rho}_1)^2$ and rearranging the factors, we end up with an equivalent form of the inclusive equations

$$\begin{cases} 2\ddot{\rho}_1 = \dfrac{\rho_1^2 - \rho_2^2 + 4y\rho_1\rho_2}{(\rho_1^2+\rho_2^2)^2} & (46a) \\ 2\dfrac{\rho_2'}{\rho_2} = \dfrac{1}{\dot{\rho}_1^2(\rho_1^2+\rho_2^2)(1+y\rho_2')} & (46b) \end{cases}$$

which is particularly suited for the expansion in ρ_2 we are going to perform.

The crucial observation is that, to each perturbative order in ρ_2, there exists a second integral of motion which allows us to express ρ_2 as a function of ρ_1 and of an "integration constant", e.g. $\rho_2(\infty) \equiv \rho_\infty$. This is easily seen by carrying out explicitly the calculation.

Lowest order By expanding in ρ_2 the r.h.s. of eqs. (46) at lowest relative order, we find for $\tau > b^2 = b_c^2$

$$\begin{cases} 2\ddot{\rho}_1 = \dfrac{1}{\rho_1^2} & (47a) \\ 2\dfrac{\rho_2'}{\rho_2} = \dfrac{1}{\dot{\rho}_1^2\,\rho_1^2}\,. & (47b) \end{cases}$$

Eq. (47a) is nothing but the exclusive equation (10) which admits the integral of motion

$$\dot{\rho}_1^2 + \frac{1}{\rho_1} = 1 \tag{48}$$

yielding $\dot{\rho}_1$ as a function of ρ_1, i.e.

$$\dot{\rho}_1 = \sqrt{1-\frac{1}{\rho_1}}\,. \tag{49}$$

Considering now eq. (47b), we can use eq. (49) to replace $\dot{\rho}_1^2$ in the r.h.s., so as to obtain the logarithmic derivative of ρ_2 with respect to ρ_1 and, after straightforward integration, ρ_2 itself:

$$2\frac{\mathrm{d}\log\rho_2}{\mathrm{d}\rho_1} = 2\frac{\rho_2'}{\rho_2} = \frac{1}{\rho_1^2\left(1-\frac{1}{\rho_1}\right)} \qquad \Longrightarrow \qquad \rho_2^2 = \rho_\infty^2\left(1-\frac{1}{\rho_1}\right)\,, \tag{50}$$

where we have taken into account the fact that $\rho_1(\infty) = \infty$ and therefore the integration constant ρ_∞ has to be identified with $\rho_2(\infty)$. The matching at $\tau = b^2 = b_c^2 = 3\sqrt{3}/2$ of the above solutions (49,50) with the free evolution (44) provides the lowest order t-values

$$t_1 = \frac{1}{\sqrt{3}} \equiv t_c\,, \qquad t_2 = \frac{2}{9}\rho_\infty\,. \tag{51}$$

First order The first order relative corrections are obtained by expanding eqs. (46) to first relative order in ρ_2:

$$\begin{cases} 2\ddot{\rho}_1 = \dfrac{1}{\rho_1^2} + 4y\dfrac{\rho_2}{\rho_1^3} & (52a) \\ \dfrac{2\rho_2'}{\rho_2} = \dfrac{1-y\rho_2'}{\dot{\rho}_1^2\,\rho_1^2} & (52b) \end{cases}$$

We now substitute the expression (50) into eq. (52a), obtaining

$$2\ddot{\rho}_1 = \frac{1}{\rho_1^2} + 4y\rho_\infty \frac{1}{\rho_1^3}\sqrt{1-\frac{1}{\rho_1}} \equiv \frac{\mathrm{d}}{\mathrm{d}\rho_1}V_1(\rho_1) \;, \tag{53}$$

whence the (first-order) conserved quantity

$$\dot{\rho}_1^2 = 1 - [V_1(\rho_1) - V_1(\infty)] = 1 - \frac{1}{\rho_1} + y\rho_\infty \frac{8}{15}\left[\sqrt{1-\frac{1}{\rho_1}}\left(2+\frac{1}{\rho_1}-\frac{3}{\rho_1^2}\right) - 2\right] \;, \tag{54}$$

where $\dot{\rho}_1(\infty) = 1$ has been imposed.

The first-order correction to ρ_2 is now found by substituting eqs. (50,54) into the r.h.s. of eq. (52b) and, after integration in ρ_1, we obtain

$$\rho_2^2 = \rho_\infty^2 \left\{1 - \frac{1}{\rho_1} + y\rho_\infty \left[\frac{8}{5} - \frac{8}{3\rho_1} + \sqrt{1-\frac{1}{\rho_1}}\left(\frac{11}{15\rho_1^2} + \frac{28}{15\rho_1} - \frac{8}{5}\right)\right]\right\} \;. \tag{55}$$

The values of t_1 and t_2 are again found from the matching at $\tau = b^2$ with the free solution (44). However, at this level of accuracy, b is slightly different from b_c and also t_1 differs from t_c. We parameterize these differences by introducing the adimensional parameters β and ϵ such that[7]

$$b^2 \equiv \frac{b_c^2}{1-\beta} \;, \qquad t_1 \equiv t_c(1+\epsilon) \;. \tag{56}$$

Note that $\beta > 0$ means $b > b_c$. Evaluating eqs. (54,39,55) at $\tau = b^2$ yields respectively

$$\beta = A_1 y t_2 \;, \qquad A_1 = \frac{4}{5}(9 - 2\sqrt{3}) \tag{57}$$

$$\epsilon = -\frac{\beta + \sqrt{3}yt_2}{3} = -\frac{1}{5}(12-\sqrt{3})yt_2 \tag{58}$$

$$\rho_\infty = \frac{9}{2}t_2\left[1 + \frac{2}{15}(27-\sqrt{3})yt_2\right] \;. \tag{59}$$

To summarize the results so far, we have found the first-order corrections to $t_1 = t_c(1+\epsilon)$, t_2 and ρ_∞, obtaining

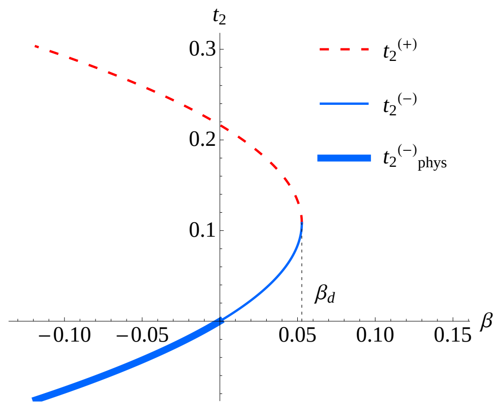
$$\beta \sim \epsilon \sim yt_2 \sim y\rho_\infty \;. \tag{60}$$

β and ϵ are of the same order, t_2 and ρ_∞ are of the same order but the latter have a y factor relative to the former. In order to deal also with the $y = 0$ case, it is then convenient to write β, ϵ and ρ_∞ in terms of t_2.

Second order We can compute the second order quantities by expanding eqs. (46):

$$\begin{cases} 2\ddot{\rho}_1 = \dfrac{1}{\rho_1^2} + 4y\dfrac{\rho_2}{\rho_1^3} - 3\dfrac{\rho_2^2}{\rho_1^4} & \text{(61a)} \\[2ex] \dfrac{2\rho_2'}{\rho_2} = \dfrac{1}{\dot{\rho}_1^2\rho_1^2}\left(1 - y\rho_2' + y^2{\rho_2'}^2 - \dfrac{\rho_2^2}{\rho_1^2}\right) \;. & \text{(61b)} \end{cases}$$

[7] This definition of β differs at $\mathcal{O}\left(\beta^2\right)$ from the analogous definition of ref. [6] eq. (72).



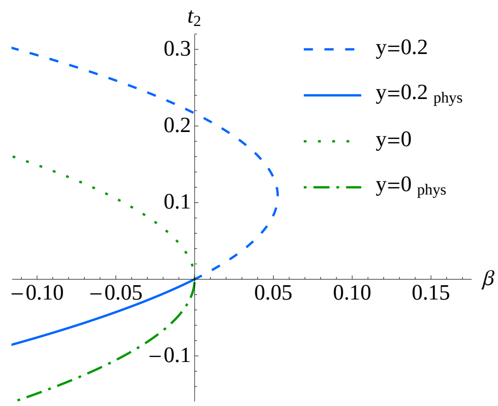


Figure 3: [Left] *The two solutions* $t_2^{(\pm)}$ *(red dashed and blue solid) versus* β *for* $y = 0.2$. *They coincide at* $b = b_d$. *Only* $t_2^{(-)}$ *for* $b < b_c$ *is physical (thick line).* [Right] *Comparison of the previous solutions for* $y = 0.2$ *(blue solid and dashed) with* $t_2^{(\pm)}$ *in the* $y = 0$ *case (green dotted and dash-dotted), the latter showing a square-root behaviour around the critical point* $\beta = 0$.

The explicit calculation is based on the strategy and the results presented at first order. Here we just write down the relations among the relevant parameters β, ρ_∞, ϵ and t_2:

$$\beta = A_1 y t_2 - \frac{9}{2}\left(1 + A_2\, y^2\right) t_2^2 \,, \qquad A_2 = \frac{291 - 112\sqrt{3}}{75} \tag{62}$$

$$\epsilon = -\frac{1}{5}(12 - \sqrt{3}) y t_2 + \frac{1}{2}\left(1 + \frac{16\sqrt{3} - 23}{25} y^2\right) t_2^2 \,. \tag{63}$$

$$\rho_\infty = \frac{9}{2} t_2 \left[1 + \frac{2}{15}(27 - \sqrt{3}) y t_2 + \left(-3 + \frac{883 - 96\sqrt{3}}{75} y^2\right) t_2^2\right] \,. \tag{64}$$

We note the different behaviour of t_2 versus β at different values of y (fig. 3b). Without opened inelastic channels ($y = 0$) the imaginary component ρ_2 of the field grows in a non-analytic way as soon as b decreases below b_c: [8]

$$\rho_\infty \sim t_2 \sim \sqrt{-\beta} \sim \sqrt{\epsilon} \,. \tag{65}$$

On the other hand, with a finite inelastic emission ($y > 0$) the imaginary component ρ_2 grows linearly for small values of $b_c - b$, i.e., $|\beta| \ll y^2 \lesssim 1$, according to eq. (57). In the intermediate region $y^2 \sim |\beta| \ll 1$, where both y and β are small, the quadratic equation (62) yields the two solutions (fig. 3a)

$$9 t_2^{(\pm)} \simeq A_1 y \pm \sqrt{A_1^2 y^2 - 18\beta} \tag{66}$$

provided its discriminant is greater than zero, and this happens only if the impact parameter b is smaller than a y-dependent critical value b_d determined by

$$\beta \leq \frac{A_1^2 y^2}{18} \equiv \beta_d(y) \,, \qquad b_d^2(y) \equiv \frac{b_c^2}{1 - \beta_d} \,. \tag{67}$$

[8] Actually, the same square-root behaviour occurs in the region $y^2 \ll |\beta| \lesssim 1$.

The above condition reproduces the structure of the complex solutions numerically found in sec. 3.2, namely the existence of the two complex solutions N_1 and N_2 for $b_c^2 < b^2 < b_d^2$. For $b > b_c$ both have positive imaginary part ($t_2^{(\pm)} > 0$), while for $b < b_c$ only one of them ($t_2^{(-)}$) becomes negative and is therefore physically acceptable according to the stability requirement.

Such different behaviours affect the inclusive action. As a consequence, there are different trends of absorption, according to whether few graviton emission is allowed ($y \to 0$) or a finite contribution of the inelastic channels is considered.

3.4 Unitarity defect

The solutions just obtained with the perturbative method allow us to compute the unitarity defect $e^{i\mathcal{A}_u}$ — at least for small enough y and β. In fact, by recalling eq. (40)

$$\mathrm{i}\mathcal{A}_u = 4\alpha \left[\rho_\infty - \frac{3}{2} \frac{t_2}{t_1^2 + t_2^2} \right] . \tag{68}$$

the imaginary part of the unitarity action is given in terms of t_1, t_2 and ρ_∞, whose dependence on the impact parameter $b \sim b_c$ has been analytically obtained in the previous section.

Above the critical point $b \geq b_c$, the physical inclusive solutions are real, $\rho_2(\tau)$ vanishes identically, and the same is true for t_2, ρ_∞ and $\mathcal{A}_u$, thus implying a unitary S-matrix. On the contrary, below the critical point $b < b_c$, the inclusive action is governed by the complex perturbative solution $t_2^{(-)} < 0$ which joins continuously with the real solution(s) at $b = b_c$. In order to study the transition across the critical point, we expand the action (68) in t_2 by means of eqs. (63) and (64), where we recall that $t_1 = (1+\epsilon)/\sqrt{3}$.

We note the interesting feature that the linear terms in t_2 cancel in the expansion of $\mathcal{A}_u$. Therefore, the suppression of the S-matrix starts at second order in t_2:

$$\frac{\mathrm{i}\mathcal{A}_u}{\alpha} = -\frac{12}{5}(9 - 2\sqrt{3}) y t_2^2 + 18 \left[1 - \frac{509 - 168\sqrt{3}}{75} y^2 \right] t_2^3 + \cdots \tag{69}$$

Due to the interplay between yt_2^2 and t_2^3, $\mathcal{A}_u$ is characterized by various regimes.

At $y = 0$ the action (69) is cubic in $t_2 \sim -\sqrt{-\beta}$:

$$\frac{\mathrm{i}\mathcal{A}_u}{\alpha} \simeq 18 t_2^3 \simeq -\frac{4\sqrt{2}}{3}(-\beta)^{3/2} = -\frac{2}{\alpha} \Im \mathcal{A}_{\mathrm{el}} , \tag{70}$$

and reproduces the fractional exponent $3/2$ of the elastic action (16) as shown in ref. [5]. Note that $\mathrm{i}\mathcal{A}_u$ is negative due to the choice of the stable solution $t_2^{(-)} < 0$. As a consequence, the S-matrix is suppressed for $b < b_c$. The same happens at small $y \ll |t_2| \sim \sqrt{|\beta|}$.

At intermediate $y \sim |t_2| \ll 1$, the two terms in eq. (69) are of the same order and the S-matrix is always suppressed for $b < b_c$ since $t_2^{(-)} < 0$.

At finite $y \gg \sqrt{|\beta|}$, the term proportional to yt_2^2 dominates the inclusive action, which shows a negative-definite quadratic behaviour in $b - b_c$, with a vanishing maximum at the critical point:

$$\frac{i\mathcal{A}_u}{\alpha} \simeq -\frac{12}{5}(9 - 2\sqrt{3}) y t_2^2 \simeq -\frac{3}{A_1} \frac{\beta^2}{y} . \tag{71}$$

This is in agreement with the numerical result represented by the solid line in fig. 2b.

The above analysis for the unitarity action can be summarized by distinguishing three regimes, at least for $y \lesssim 1$:

- For $b < b_c$ and $(b - b_c)^2 \gg y$, the unitarity action $\mathrm{i}\mathcal{A}_u$ is negative and decreases in size with a power-like behaviour $(b_c - b)^{3/2}$ as $b \to b_c$. This includes the case $y = 0$.
- For $b < b_c$ and $(b - b_c)^2 \ll y$, the unitarity action $\mathrm{i}\mathcal{A}_u$ is negative and vanishes quadratically at $b = b_c$.
- For $b \geq b_c$ the unitarity action vanishes identically.

It is important to realize that the proper choice of the physical solution avoids a unitarity excess $\langle 0|S^\dagger S|0\rangle > 1$. Such an unphysical behaviour would occur if we don't reject the $t_2^{(+)}$ solution, since the corresponding inclusive action $\mathrm{i}\mathcal{A}_u(t_2^{(+)})$ would be positive for $b \lesssim b_c$, as shown by the circles in fig. 2b.

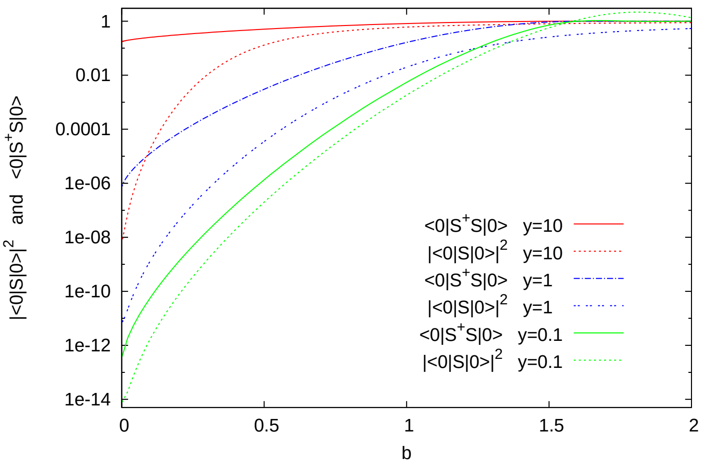


Figure 4: *Numerical evaluation of the unitarity deficit (solid lines) at various values of y, and comparison with the quantum v.e.v. squared of the S-matrix (dashed lines) illustrating the increasing contribution of the inelastic channels at larger y.*

The numerical evaluation of the ensuing S-matrix suppression for various values of y is presented in fig. 4, where we plot the semiclassical estimate of the v.e.v. $\langle 0|S^\dagger S|0\rangle = \mathrm{e}^{\mathrm{i}\mathcal{A}_u}$ and compare it with the $n = 0$ vacuum contribution $|\langle n = 0|S|0\rangle|^2$ of the unitarity sum (34). The results show that, if y is small, there is a considerable unitarity defect in the S-matrix, the inelastic channels providing only a small correction to the elastic suppression. At larger values of y there is an important recovery of unitarity due to the contribution of the inelastic channels. However, it must be kept in mind that y — related to the maximum rapidity of the emitted gravitons — cannot be arbitrarily large, because energy conservation limits the energy of the emitted particles to be smaller than the available energy $\sqrt{s}$. Actually, energy conservation prevents y to assume large values, which at most can be of order $y \sim \mathcal{O}(1)$.

In conclusion, a unitary defect is always present in the semiclassical estimate of $\langle 0|S^\dagger S|0\rangle$ when the impact parameter b is smaller than the critical one b_c. In this region the contribution of inelastic processes only partially compensates the suppression of the elastic channel, at least at semiclassical level.

4 Search for quantum transitions

The semiclassical estimate of unitarity of the S-matrix performed in the previous section gave an exponential suppression of $\langle 0|S^\dagger S|0\rangle \sim \mathrm{e}^{-Gs}$ for $b < b_c$. In this section we investigate whether quantum effects provide larger contributions eventually restoring unitarity. The quantity that we are going to study at quantum level is the eigenvalue of the S-matrix operator (19). This is possible because the specific form of the S-matrix — a coherent-state operator acting on the Fock space of gravitons. — allows us to determine its eigenvectors and eigenvalues [7], as we briefly recall.

In order to construct such eigenvectors, it is convenient to introduce (normalized) graviton-coherent-states which are parameterized by arbitrary (complex) profile functions $\eta(\tau)$ as follows:

$$|\eta(\tau)\rangle \equiv \mathrm{e}^{-\frac{1}{2}(\eta^*,\eta)}\mathrm{e}^{(\eta^*,a^\dagger)}|0\rangle \;, \tag{72}$$

with the short-hand notation $(\eta,\zeta) \equiv \int_0^\infty \eta(\tau)\zeta(\tau)\,\mathrm{d}\tau$ in terms of the azimuthally-averaged annihilation operators of eq. (6)

$$a(\tau = \boldsymbol{x}^2) \equiv \int_0^{2\pi} \frac{\mathrm{d}\phi_{\boldsymbol{x}}}{2\sqrt{\pi}}\,\frac{A(\boldsymbol{x})}{\sqrt{Y}} \qquad \Rightarrow \qquad [a(\tau), a^\dagger(\tau')] = \delta(\tau-\tau') \;. \tag{73}$$

Since the action of the S-matrix operator on those states is a superposition of coherent states with shifted parameter

$$S|\eta\rangle = \int [\mathcal{D}\rho]\;\mathrm{e}^{-\mathrm{i}\int L(\rho)\,\mathrm{d}\tau}\,|\eta + \mathrm{i}\delta_\rho\rangle \;, \qquad \left(\delta_\rho \equiv \sqrt{2\alpha y}(1-\dot{\rho})\right) \;, \tag{74}$$

the functional Fourier transform of coherent states with imaginary parameter

$$|\{\omega(\tau)\}\rangle \equiv \mathrm{e}^{\frac{1}{4}(\omega,\omega)} \int [\mathcal{D}\zeta(\tau)]\;\mathrm{e}^{-\mathrm{i}(\zeta,\omega)}\,|\mathrm{i}\zeta(\tau)\rangle \tag{75}$$

constitutes a complete set of S-matrix eigenstates. With the pre-factor $\mathrm{e}^{\frac{1}{4}(\omega,\omega)}$ such states are normalized according to the delta functional

$$\langle\{\omega'(\tau)\}|\{\omega(\tau)\}\rangle = \delta(\{\omega-\omega'\}) \;. \tag{76}$$

By acting on the states (75) with the S-matrix (19), it is straightforward to verify that the former are eigenstates of the latter. The corresponding eigenvalues, that we denote with $\mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]}$, obviously depend on the real functional parameter $\omega(\tau)$, and are given by the path-integral

$$\mathrm{eigenv}_\omega(S) \equiv \mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]} = \int_{\rho(0)=0}^{\dot{\rho}(\infty)=1} [\mathcal{D}\rho]\;\mathrm{e}^{-\mathrm{i}\int L(\rho)+\mathrm{i}(\delta_\rho,\omega)} \;. \tag{77}$$

Endowed with eigenstates and eigenvalues of the S-matrix, we can perform a quantitative study of its unitarity properties at quantum level. For instance, we can reconsider the v.e.v. of $S^\dagger S$ by inserting the completeness of ω states, obtaining

$$\langle 0|S^\dagger S|0\rangle = \int [\mathcal{D}\omega]\;\langle 0|S^\dagger|\{\omega\}\rangle\langle\{\omega\}|S|0\rangle = \int [\mathcal{D}\omega]\;\mathrm{e}^{-\frac{1}{2}(\omega,\omega)}\mathrm{e}^{-2\Im\mathcal{A}[\omega]} \;. \tag{78}$$

In the r.h.s. of the previous expression, we note two elements that determine the order of magnitude of the v.e.v.: the "density" of the intermediate state (ω,ω) and the eigenvalue

(actually, its modulus squared). If we find $|\{\omega\}\rangle$ states such that $(\omega, \omega) = \mathcal{O}(1)$ and $\Im\mathcal{A}[\omega]$ is much smaller than the tunneling exponent, we have a case for an important quantum effect, reducing the unitarity defect.

At semiclassical level, the eigenvalue (77) is found from the contribution of ρ which makes the action

$$\mathcal{A}[\omega] \equiv -\mathrm{i} \int \left[L(\rho, \dot{\rho}, \tau) - \sqrt{2\alpha y}(1 - \dot{\rho})\omega \right] \, \mathrm{d}\tau \tag{79}$$

stationary. The ensuing Euler-Lagrange equation is similar to that of the elastic amplitude (10), but with an additional term representing an "external force" proportional to $\mathrm{d}\omega/\mathrm{d}\tau$ which depends on the eigenstate:

$$\ddot{\rho} - \frac{\Theta(\tau - b^2)}{2\rho^2} = -\sqrt{\frac{2y}{\alpha}}\dot{\omega}(\tau) \; . \tag{80}$$

It is this external force that might help to cross the repulsive Coulomb barrier thus avoiding the exponential suppression for proper values of $\omega(\tau)$.

Here, however, we want to estimate the eigenvalue (77) at quantum level. In order to do that, we note that $\mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]}$ can be rewritten as a matrix element — in the Hilbert-space quantizing the 1D system (80) — of a proper evolution operator $\mathcal{U}_\omega(0, \infty)$, in complete analogy to the quantum analysis of the elastic amplitude of sec. 2.3:

$$\mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]} = \langle \rho = 0 | \mathcal{U}_\omega(0, \infty) | \dot{\rho} = 1 \rangle \; . \tag{81}$$

The bra and ket states embody the boundary conditions of the functional integration, while the Möller operator $\mathcal{U}_\omega$ governs the $\omega(\tau)$-dependent dynamics, and is determined in the following way. Starting from the effective lagrangian defined by the integrand in eq. (79), by Legendre transform we derive the conjugate momentum and the hamiltonian

$$\Pi \equiv \frac{\partial L_\omega}{\partial \dot{\rho}} = 2\alpha(\dot{\rho} - 1) + \sqrt{2\alpha y}\,\omega(\tau) \tag{82}$$

$$H_\omega = \alpha \left[\left(\frac{\Pi + 2\alpha - \sqrt{2\alpha y}\,\omega}{2\alpha} \right)^2 + \frac{\Theta(\tau - b^2)}{\rho} \right] + \sqrt{2\alpha y}\,\omega \; . \tag{83}$$

We quantize this system by imposing canonical commutation relations $[\rho, \Pi] = \mathrm{i}$, e.g., by identifying the operator $\Pi + 2\alpha = -\mathrm{i}\partial/\partial\rho$ in the ρ-coordinate representation. Finally, the evolution operator $\mathcal{U}_\omega$ obeys the differential equation

$$\mathrm{i}\frac{\partial}{\partial\tau}\mathcal{U}_\omega(\tau, \infty) = H_\omega \mathcal{U}_\omega \; . \tag{84}$$

By defining the state

$$|\psi_\omega(\tau)\rangle \equiv \mathcal{U}_\omega(\tau, \infty)|\dot{\rho} = 1\rangle \qquad \text{obeying} \qquad \mathrm{i}\frac{\partial}{\partial\tau}|\psi_\omega\rangle = H_\omega|\psi_\omega\rangle \; , \tag{85}$$

we can express the S-matrix eigenvalue (81) as the amplitude of the wave function at $\rho = 0$ and $\tau = 0$:

$$\psi_\omega(\rho, \tau) \equiv \langle \rho | \psi_\omega(\tau) \rangle \qquad \Longrightarrow \qquad \mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]} = \psi_\omega(0, 0) \; . \tag{86}$$

The above wave function can be determined by solving the Schrödinger equation (85b). Before doing that, we eliminate the $\omega(\tau)$-dependent shift in momentum and energy by the similarity transformation

$$\psi_\omega(\rho,\tau) \to \mathrm{e}^{\mathrm{i}\sqrt{2\alpha y}\,\omega(\tau)\rho}\,\mathrm{e}^{\mathrm{i}\sqrt{2\alpha y}\int_\tau^\infty \omega(\tau')\,\mathrm{d}\tau'}\,\Psi_\omega(\rho,\tau) \tag{87}$$

so that the Schrödinger equation for Ψ is

$$\mathrm{i}\frac{\partial}{\partial\tau}\Psi(\rho,\tau) = \left[\alpha\left(-\frac{1}{4\alpha^2}\frac{\partial^2}{\partial\rho^2} - 1 + \frac{\Theta(\tau-b^2)}{\rho}\right) + \sqrt{2\alpha y}\,\rho\,\dot{\omega}(\tau)\right]\Psi(\rho,\tau) \tag{88}$$

The shift in momentum has produced a linear potential term due to a τ-dependent external force, as already noted before eq. (80). Since ψ and Ψ differ only by an unimportant phase factor, we can replace the former with the latter in eq. (86) and use the hamiltonian in eq. (88) to compute the time-evolution operator $\mathcal{U}_\omega$.

Let us now compute $\Psi(0,0)$ in the case $b=0$: in this way the τ-dependence only comes from the external force $\dot{\omega}$. We adopt the perturbative approach by splitting H_ω in an unperturbed term $H_{\omega=0} = H_c$ which coincides with the "Coulomb" hamiltonian (29) governing the elastic amplitude, and in a perturbation given by the time-dependent potential $\sqrt{2\alpha y}\dot{\omega}\rho$. For the sake of simplicity, we assume that the external force is active only within a finite time range $0<\tau<T$ so that for $\tau>T$ the evolution is just of Coulomb type. In this way the wave function can be determined by expanding the evolution operator $\mathcal{U}_\omega$ in Dyson's series, yielding at first order

$$\Psi_\omega(\rho,0) = \psi_c(\rho) - \mathrm{i}\sqrt{2\alpha y}\int_0^T \mathrm{d}\tau\;\dot{\omega}(\tau)\,\langle\rho|\mathrm{e}^{\mathrm{i}H_c\tau}\rho\,\mathrm{e}^{-\mathrm{i}H_c\tau}\,|\psi_c\rangle + \cdots\,, \tag{89}$$

We obtain a more explicit expression by inserting a complete set of unperturbed eigenstates $|\phi^{(n)}\rangle$ of energy E_n before the operator ρ, by exploiting eq. (29), i.e., $H_c|\psi_c\rangle = 0$, and then projecting on the position eigenstate $\langle\rho=0|$:

$$\mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]}|_{b=0} \simeq \psi_c(0) - \mathrm{i}\sqrt{2\alpha y}\,\sum_n \phi^{(n)}(0)\,\langle\phi^{(n)}|\rho|\psi_c\rangle\int_0^T \mathrm{e}^{\mathrm{i}E_n\tau}\dot{\omega}(\tau)\,\mathrm{d}\tau\,. \tag{90}$$

We see that the effect of the interaction potential $\sim\dot{\omega}\rho$ is to cause transitions from the initial state ψ_c towards other eigenstates of H_c. Eq. (90) can then be interpreted by saying that quantum effects provide a recovery of unitarity if the interaction potential induces transitions towards states with a wave function at the origin much larger than the tunneling amplitude: $\phi_n(0) \gg \psi_c(0) \sim \mathrm{e}^{-\pi\alpha}$. We expect that such states could belong to two groups:

- bound states whose wave function is significantly different from zero in the region $\rho<0$ and is finite at $\rho=0$;
- high-energy continuum states for which the suppression from the Coulomb barrier is smaller.

More precisely, the entity of the contribution of those states to the eigenvalue $\mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]}$ in eq. (90) is given by three factors: the wave function at the origin $\phi_n(0)$, the matrix element $\langle\phi_n\rho|\psi_c\rangle$ and the time integral. The first two will be studied in detail in the next sections.

As for the time integral, we need to specify the function $\omega(\tau)$. By analogy with the quantum mechanical phenomenon of induced transitions, we consider as a good candidate of external force an oscillatory function which can induce big transition amplitude at resonance. More precisely we choose

$$\omega(\tau) = \frac{1}{\sqrt{T}} \sin(K\tau)\Theta(T-\tau) \, , \tag{91}$$

where the normalization has been chosen to keep $(\omega, \omega) = \mathcal{O}\,(1)$ in order to avoid the exponential suppression in eq. (78). Whith this choice, the time-factor reads

$$\mathcal{T}_n \equiv \int_0^T \mathrm{e}^{\mathrm{i}E_n\tau} \dot{\omega}(\tau)\, \mathrm{d}\tau = \frac{K}{2\sqrt{T}} \left[\frac{\mathrm{e}^{\mathrm{i}(E_n-K)T} - 1}{\mathrm{i}(E_n - K)} + (K \to -K) \right] \, . \tag{92}$$

We are interested in situations where the perturbative corrections to the eigenvalue (90) are larger than the lowest order term. In this case, the square modulus of the eigenvalue can be written as

$$\left|\mathrm{e}^{\mathrm{i}\mathcal{A}[\omega]}\right|^2 \simeq \sum_{n,n'} c_n c_{n'}^* \mathcal{T}_n \mathcal{T}_{n'}^* \simeq K^2 \sum_{n,n'} c_n c_{n'}^* \mathrm{e}^{\mathrm{i}(E_n - E_{n'})T} \frac{\sin[(E_n - K)T]\sin[(E_{n'} - K)T]}{(E_n - K)(E_{n'} - K)T} \, , \tag{93}$$

where the c_n's contain the ρ matrix elements and the wave function at the origin. For long interaction time T, the preceding expression is peaked for $E_n = K$ and provides a delta function of energy conservation $K^2\delta(E_n - K)$ which suppresses the interference terms with respect to the squares of the individual amplitues. The latter provide a contribution of order $\mathcal{O}\,(1)$ around the states compatible with conservation. Therefore, the time factor is $\mathcal{O}\,(1)$ if $(\omega, \omega) = \mathcal{O}\,(1)$.

4.1 Transitions to bound states

According to eq. (90), the entity of quantum effects to unitarity depends crucially on the matrix elements $\langle \phi_n | \rho | \psi_c \rangle$ and on the wave function at the origin $\phi_n(0)$ of the state reached by the induced transition. In this section we determine both factors for the case of transitions to bound states.

In fact, the Coulomb potential $\sim 1/\rho$ in the unperturbed hamiltonian H_c is attractive for $\rho < 0$ giving rise to a discrete spectrum associated to bound states which could correspond to collapsed states characterized by a strong gravitational field $h \propto (1 - \dot{\rho})$. The discrete spectrum of H_c is given by the "energy" eigenvalues

$$E_n = -\alpha - \frac{\alpha^3}{(n + \frac{1}{2})^2} \qquad (n \in \mathbb{N}) \tag{94}$$

and the (normalized) eigenfunctions are expressed in terms of the irregular Whittaker function [16] W (closely related to the confluent hypergeometric function U)

$$\phi_n(\rho) = N_n \left[\Gamma\left(\tfrac{3}{2} + n\right) W_{-(n+\frac{1}{2}),\frac{1}{2}}(x)\; \Theta(x) + \Gamma\left(\tfrac{3}{2} - n\right) W_{n+\frac{1}{2},\frac{1}{2}}(-x)\; \Theta(-x) \right] \tag{95}$$

$$N_n = \phi_n(0) = \frac{c\,\alpha}{(n + \frac{1}{2})^{3/2}} \, , \qquad x \equiv \frac{4\alpha^2 \rho}{n + \frac{1}{2}} \tag{96}$$

where $c \simeq 0.45$. These wave functions are significantly different from zero for $\rho < 0$, and in particular where the energy is larger than the potential (see fig. 5). At variance with $\psi_c(0) \sim \mathrm{e}^{-\pi\alpha}$, their value at the origin $\phi_n(0)$ is not suppressed with α, but only by a power of n, as can be read from eq. (96). For this reason, transitions to bound states might be important for the estimate of the eigenvalue (90). The crucial factor is then the matrix element

$$\langle \phi_n | \rho | \psi_c \rangle = \int_{-\infty}^{+\infty} \phi_n^*(\rho) \rho \, \psi_c(\rho) \, \mathrm{d}\rho \tag{97}$$

which can be estimated by using the WKB approximations for ϕ_n and ψ_c. We use the notation $\tilde{n} \equiv n + \frac{1}{2}$ and divide the integration domain into 3 regions:

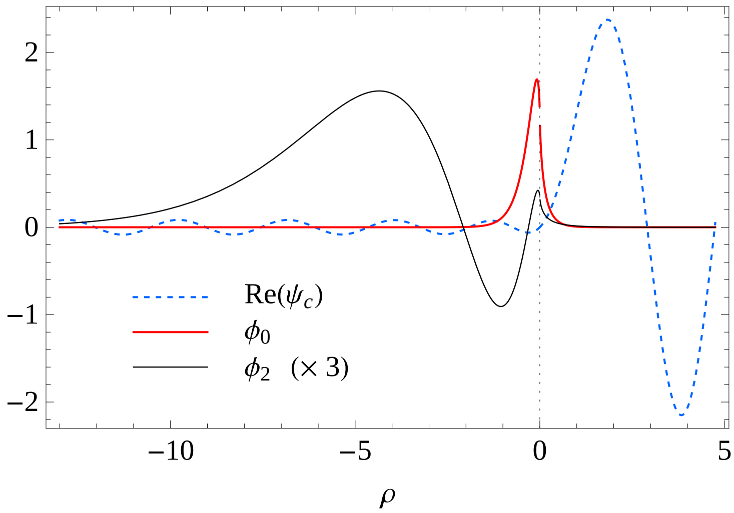


Figure 5: *Plot of* $\Re\psi_c$ *(dashed) and of* ϕ_n *(solid) for* $n = 0$ *(thick) and* $n = 2$ *(thin), the last function being multiplied by a factor of 3. Here* $\alpha = 1$ *is small in order not to have a huge suppression of the wave functions in the respective classically forbidden regions.*

($\boldsymbol{\rho < 0}$) For negative ρ, ψ_c oscillates with amplitude $\simeq \mathrm{e}^{-\pi\alpha}$ because of the tunneling suppression. The generic bound state eigenfunction oscillates in the interval $-(\tilde{n}/\alpha)^2 < \rho < 0$, reaching its maximum value in the leftmost half-period, with amplitude $\sim \alpha\tilde{n}^{-7/3}$, and then decreases exponentially towards zero for $\rho \to -\infty$. Therefore, the convergent oscillatory integrand is uniformly bounded by $\alpha^{-1}\tilde{n}^{-1/3}\mathrm{e}^{-\pi\alpha}$ and the ensuing contribution to the matrix element is exponentially suppressed in α.

($\boldsymbol{\rho > 1}$) In this region, ψ_c is an oscillating function with maximum amplitude $\sim \alpha^{1/6}$. On the other hand, $|\phi_n| < N_n \Gamma(1+\tilde{n})\tilde{n}^{-\tilde{n}}\mathrm{e}^{-2\alpha^2\rho/\tilde{n}}$ is strongly suppressed. It turns out that the product $\phi_n\psi_c$ is oscillatory and exponentially suppressed in α^2, and so is the contribution of this region to the matrix element (97).

($\boldsymbol{0 < \rho < 1}$) In the intermediate region $|\psi_c|$ increases from the value $|\psi_c(0)| \simeq \mathrm{e}^{-\pi\alpha}$ to values of order one, like $\mathrm{e}^{-\pi\alpha}\mathrm{e}^{4\alpha\sqrt{\rho}}$, while ϕ_n starts from $\phi_n(0) \sim \alpha/\tilde{n}^{3/2}$ and goes to zero as *(i)* $N_n \mathrm{e}^{-4\alpha\sqrt{\rho}}$ if $\rho \ll (\tilde{n}/\alpha)^2$; *(ii)* $N_n\Gamma(1+\tilde{n})(x+\tilde{n})^{-\tilde{n}}\mathrm{e}^{-2\alpha^2\rho/\tilde{n}}$ if $\rho \gtrsim (\tilde{n}/\alpha)^2$. It turns out that the product of wave functions $\phi_n\psi_c$ is also in this case exponentially suppressed.

In fact, in case *(i)*, the two esponential with $\sqrt{\rho}$ cancel out and the product is of order $\mathrm{e}^{-\pi\alpha}$; in case *(ii)* we have

$$|\phi_n\psi_c| \lesssim \mathrm{e}^{-\pi\alpha} N_n \sqrt{2\pi\tilde{n}}\mathrm{e}^{-\tilde{n}}\mathrm{e}^{f(x,\tilde{n})} \ , \quad f(x,\tilde{n}) \equiv -\frac{x}{2} + 2\sqrt{\tilde{n}x} - \tilde{n}\ln\left(1+\frac{x}{\tilde{n}}\right) \ , \quad \tilde{n} \equiv n + \frac{1}{2} \ . \tag{98}$$

The exponent $f(x,\tilde{n})$ has a maximum at $x = \tilde{n}$, with value $f_{\max} = (\frac{3}{2} - \ln 2)\tilde{n}$. Therefore, the integrand is uniformly bounded by a (decreasing) function of n times $\alpha\mathrm{e}^{-\pi\alpha}$. Therefore also in this intermediate region the matrix element is exponentially suppressed in α.

These results are confirmed by numerically computing the matrix element (97), as shown in fig. 6. In conclusion, at least in this first-order perturbative treatment, the S-matrix eigenvalue (90) does not receive significative enhancements from the bound-states eigenfunctions beyond the exponentially suppressed contribution $\psi_c(0)$.

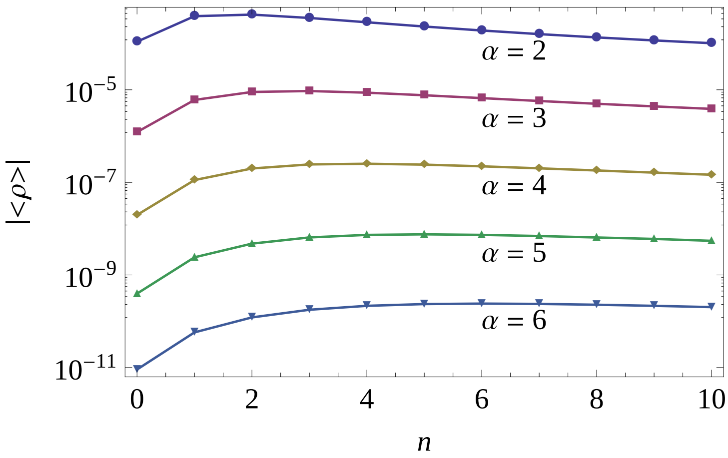


Figure 6: *Numerical computation of the matrix element (97) for n up to 10. Each line represents a different value of α, here ranging from 2 to 6.*

4.2 Transitions to continuum states

The second mechanism of possible restoration of unitarity that we consider is the transition towards "high-energy" eigenstates $\langle\phi_E|$ of the continuum spectrum of H_c. The motivation is that, for $\rho \leq 0$, we expect the corresponding eigenfunctions $\phi_E(\rho)$ to be less suppressed than ψ_c, which is just the zero-energy eigenfunction. The eigenfunctions obey the Schrödinger equation ($\hbar = 1$)

$$H_c\,\phi_E \equiv \left[-\frac{1}{4\alpha}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \alpha\left(\frac{1}{\rho} - 1\right)\right]\phi_E = E\,\phi_E \tag{99}$$

and the condition of having only transmitted wave for $\rho < 0$. If we write the energy $E = \alpha(t_0^2 - 1)$ in terms of the parameter $t_0 \geq 0$, it is easy to check that, with the rescalings $\alpha \to \alpha/t_0$ and $\rho \to \rho t_0^2$, eq. (99) reduces to eq. (29) for $\psi_c = \phi_0$. Therefore, the generic eigenfunction can be expressed in terms of ψ_c, namely

$$\phi_E(\rho;\alpha) = \sqrt{\frac{\alpha}{\pi}}\,\psi_c\big(t_0^2\rho;\frac{\alpha}{t_0}\big) \ , \tag{100}$$

where the prefactor $\sqrt{\alpha/\pi}$ has been chosen in order to obtain the continuum normalization

$$\int \phi_E^*(\rho)\, \phi_{E'}(\rho)\, \mathrm{d}\rho = \delta(t_0 - t_0') \; . \tag{101}$$

Eq. (100) clearly shows that, for large energies $E \gg \alpha \iff t_0 \gg 1$, the wave function at the origin $\phi_E(0) \sim \mathrm{e}^{-\frac{\pi\alpha}{t_0}}$ is really much larger than $\psi_c(0)$, because it is easier to cross the barrier at $\rho \geq 0$.

It remains to evaluate the matrix elements $\langle \phi_E | \rho | \psi_c \rangle$. By using the integral representation of ϕ_c derived in ref. [7] and the relation (100), we can write

$$\phi_E(\rho) = \sqrt{\frac{\alpha}{\pi}} N\big(\frac{\alpha}{t_0}\big) \int_{-\infty-\mathrm{i}0}^{+\infty} \mathrm{sign}(t-1) \left(\frac{t-1}{t+1}\right)^{\mathrm{i}\frac{\alpha}{t_0}} \frac{\mathrm{e}^{\mathrm{i}2\alpha\rho t t_0}}{t^2-1}\, \mathrm{d}t \tag{102}$$

$$N(\alpha) \equiv \frac{\mathrm{e}^{-\frac{\pi\alpha}{2}}}{\cosh(\pi\alpha)\Gamma(\mathrm{i}\alpha)} \; . \tag{103}$$

With this representation we can explicitly perform the ρ integration in the matrix element, as outlined in app. A. The result is expressed in terms of hypergeometric functions:

$$\int_{-\infty}^{+\infty} \phi_E^*(\rho)\rho\psi_c(\rho)\, \mathrm{d}\rho = N(\alpha) N^*\big(\frac{\alpha}{t_0}\big) \frac{\pi}{4\alpha^2\alpha_0} \frac{\partial^2}{\partial t_0^2} I(t_0) \tag{104}$$

$$\begin{aligned} I(t_0) = \left(\frac{t_0-1}{t_0+1}\right)^{\mathrm{i}(\alpha+\alpha_0)} & \mathrm{e}^{\pi\alpha_0} \Big[\pi\alpha_0\zeta \coth(\pi\alpha) F(1-\mathrm{i}\alpha, 1-\mathrm{i}\alpha_0; 2; \zeta) \\ & + \frac{\Gamma(-\mathrm{i}\alpha)\Gamma(1-\mathrm{i}\alpha_0)}{\Gamma\big(1-\mathrm{i}(\alpha+\alpha_0)\big)} F\big(-\mathrm{i}\alpha, -\mathrm{i}\alpha_0; 1-\mathrm{i}(\alpha+\alpha_0); 1-\zeta\big) \Big] \; , \end{aligned} \tag{105}$$

where $\zeta \equiv -\frac{4t_0}{(1-t_0)^2}$ and $\alpha_0 \equiv \frac{\alpha}{t_0}$.

Let us now study the order of magnitude of $I(t_0)$ which determines the matrix element to continuum states. We firstly consider transitions to high-energy states ($t_0^2 = 1 + \frac{E}{\alpha} \gg 1$). In this case, the variable $\zeta \simeq -\frac{4}{t_0}$ in $I(t_0)$ tends to zero, so that the two hypergeometric functions assume finite values and cannot give exponentially enhanced contributions. Therefore, the order of magnitude of $I(t_0)$ is determined by the factor $\mathrm{e}^{\pi\alpha_0}$ in eq. (105). By taking into account the wave-function normalizations, the matrix element is of order

$$N(\alpha) N^*\big(\frac{\alpha}{t_0}\big) I(t_0) \sim \frac{\mathrm{e}^{-\frac{\pi\alpha}{2}} \mathrm{e}^{-\frac{\pi\alpha_0}{2}}}{\cosh(\pi\alpha)\cosh(\pi\alpha_0)\Gamma(\mathrm{i}\alpha)\Gamma(\mathrm{i}\alpha_0)} \mathrm{e}^{\pi\alpha_0} \sim \mathrm{e}^{-\pi\alpha} \tag{106}$$

This result shows that the matrix element for transitions to continuum states of very high energy suffers a suppression comparable to that of tunneling. As a consequence, unitarity cannot be restored by these quantum effects.

Secondly, we consider transitions to lower energy states characterized by $t_0 \sim 1$ but $\alpha - \alpha_0 = \frac{\alpha}{t_0}(t_0 - 1) \gg 1$, corresponding to a quite large jump in energy, implying a lower suppression of $\phi_E(0) \sim \mathrm{e}^{-\pi\alpha_0}$. In this regime, $\zeta \to -\infty$ and the corresponding behaviour of the hypergeometric function is

$$\begin{aligned} F(1-\mathrm{i}\alpha, 1-\mathrm{i}\alpha_0; 2; \zeta) &\sim \mathrm{e}^{\pi\alpha_0} \\ F\big(-\mathrm{i}\alpha, -\mathrm{i}\alpha_0; 1-\mathrm{i}(\alpha+\alpha_0); 1-\zeta\big) &\sim A_1 \mathrm{e}^{-\pi(\alpha-\alpha_0)} \mathrm{e}^{-\pi\alpha} + A_2 \mathrm{e}^{-\pi\alpha_0} \; . \end{aligned}$$

The leading term is provided by the first hypergeometric function, yielding $I(t_0) \sim \mathrm{e}^{2\pi\alpha_0}$. By inserting the wave-function normalization factors (of order $\mathrm{e}^{-\pi(\alpha+\alpha_0)}$), the tunneling amplitude

$$\phi_E(0)\langle\phi_E|\rho|\psi_c\rangle \sim \mathrm{e}^{-\pi\alpha_0}\mathrm{e}^{-\pi(\alpha-\alpha_0)} \sim \mathrm{e}^{-\pi\alpha} \tag{107}$$

turns out to be exponentially suppressed also in this case.

In conclusion, the first-order perturbative contribution to the eigenvalue (90) are exponentially suppressed by the matrix element in all those cases where we expected sizeable effects thanks to the enhancement of the eigenfuncions at the origin. Therefore, the quantum effects do not modify the semiclassical picture described in sec. 3.

5 $\rho(0)$-fluctuations and short-distance singularities

In the preceding sections we have definitely confirmed the unitarity defect previously found at semiclassical level for $b < b_c$ [7]: indeed, the critical behaviour of the action on the physical solutions around $b \simeq b_c$ is as expected (sec. 3) and the extra-contributions to the S-matrix elements due to quantum transitions are likewise suppressed (sec. 4). However, we have kept throughout the analysis the ACV boundary conditions $\dot\rho(\infty) = 1$ and $\rho(0) = 0$. While the former is needed in order to match the large-distance behaviour with perturbative gravity and is thus unavoidable, the latter insures that the solutions being considered are UV-safe, so as to make the effective action self-sufficient. Is this really required? What happens if we let $\rho(0)$ fluctuate?

Let us recall that a non-vanishing value of $\rho(0) \equiv \rho_0$ (of order R^2) implies singularities at $r^2 = 0$ of $\dot\phi(r^2)$ and of $h(r^2)$, as follows

$$(2\pi)^2\dot\phi \simeq -\frac{\rho_0}{r^2}\,, \qquad h = \nabla^2\phi \simeq -\frac{\rho_0}{\pi^2}\delta(r^2)\,, \tag{108}$$

where the latter (implied by the outgoing flux of $\nabla\phi$) can be interpreted by assigning $\rho(r^2)$ a discontinuity ρ_0 at $r^2 = 0$ so as to have

$$(\pi R)^2 h(r^2) = (1-\dot\rho)_{\mathrm{reg}} - \rho_0 R^2\delta(r^2)\,. \tag{109}$$

Although the form of such singularities is probably model dependent, their existence is expected for generic boundary conditions. They stem from the very fact that R^2 is the dimensionful coupling of the 2-dimensional action (3) and in fact they occurred already in the more general 2-dimensional treatment of [4].

The above singularities produce a singular metric (12) and a divergent action. If we take them seriously, the divergent part of the action in eq. (19) is

$$\mathcal{A} = -\alpha\int_0^\infty\left[(1-\dot\rho)^2 - \frac{R^2}{\rho}\Theta(r^2-b^2)\right]\frac{\mathrm{d}r^2}{R^2} + \sqrt{2\alpha y}\int_0^\infty(1-\dot\rho)[a(r^2)+a^\dagger(r^2)]\,\frac{\mathrm{d}r^2}{R} \tag{110}$$

$$= \mathcal{A}_{\mathrm{div}} + \mathcal{A}_{\mathrm{reg}}\,, \qquad \mathcal{A}_{\mathrm{div}}(\rho_0) \simeq -\alpha\rho_0^2\delta(0) - \sqrt{2\alpha y\rho_0\delta(0)}\,(a_0 + a_0^\dagger)\,, \tag{111}$$

where the formally divergent $\delta(0) \simeq \mathcal{O}\left(\frac{R^2}{\lambda_s^2}\right)$ is presumably regularized by the string in the ACV approach. Note that the divergence affects not only the c-number part of $\mathcal{A}$, but also the $r^2 = 0$ mode of the coherent state operator, that we have normalized to $[a_0, a_0^\dagger] = 1$.

At this point we notice that the divergent part of the S-matrix suppresses in a drastic way elastic and quasi-elastic processes with $\rho_0 \neq 0$ because

$$\langle 0|\mathrm{e}^{\mathrm{i}\mathcal{A}_{\mathrm{div}}(\rho_0)}|0\rangle \sim \mathrm{e}^{-\alpha(\mathrm{i}+y)\rho_0^2\delta(0)} \simeq \mathrm{e}^{-\alpha(\mathrm{i}+y)\rho_0^2\frac{R^2}{\lambda_s^2}} \tag{112}$$

shows violent oscillations and absorption, with exponent of order $\sim \alpha\rho_0^2\frac{R^2}{\lambda_P^2} \sim \alpha^2\rho_0^2$ for $\lambda_s \sim \lambda_P = \sqrt{G\hbar}$. Therefore, at quasi-elastic level, we were justified in setting $\rho_0 = 0$ in the first place. And that would close the argument.

The above conclusion is not fully satisfactory, though. On one hand, $\rho_0 \neq 0$ is associated to a singular metric that we think corresponds to classically trapped solutions [17–21], and we would like to know about their fate at quantum level. On the other hand, if we think of the action (110) from the unitarity point of view, the divergent part is, after all, a hermitian operator and we expect it to contribute a unitary S-matrix if we sum over all possible emissions. This statement is confirmed by the inclusive equations (sec. 3) because we do have real-valued solutions for $b < b_c$ also, provided we take $\rho_0 = \tilde{\rho}_0 = \rho_m(b) > 0$, as mentioned at the end of sec. 2.2. In this cases, the inclusive action vanishes: should we conclude that considering $\rho_0 \neq 0$ insures S-matrix unitarity of the model?

The trouble with our second argument is that it is inconsistent with energy conservation. In fact, the action (110) provides total probabilities of order unity only in association with quite a number $\sim \alpha\frac{R^2}{\lambda_s^2}$ of hard emitted gravitons with energies of the order of the Planck mass. Barring other dynamical effects (possibly providing a red-shift), this would require an energy which is much larger than $\sqrt{s}$.

Nevertheless, from the above considerations, our tentative conclusion is that, from the unitarity stand point, we *should* actually consider the solutions with $\rho_0 \neq 0$ also by keeping in mind that the present model is inadequate in the short-distance region and we should therefore come back to the string dynamics, that was originally neglected in the regime $R \gg \lambda_s$. The role of the latter is not only that of regularizing the would-be singularities at the string scale, but also of providing a number of physical effects at distances which are intermediate between λ_s and R.

For instance, a well-known [1, 22] effect, occurring already at distances of order R, is string excitation by tidal forces (or "diffractive" string excitation). It looks straightforward to estimate it in the present model, but it has not been done yet. Another issue, perhaps more important for the unitarity problem, is the evaluation of rescattering corrections [4]. Here the string is needed in order to regulate the UV divergencies in the longitudinal variables so as to extract a finite answer. We expect from them an improvement of the longitudinal dynamics[9] for $b = \mathcal{O}(R)$ and perhaps some hint as to the existence of trapped solutions for $\rho_0 \neq 0$. If effects of this kind turn out to yield an important contribution to unitarity, then this will involve relatively soft gravitons, of energies $\sim \hbar/R$ and, for this reason, may be consistent with energy conservation.

It may be also that string effects are mostly confined at scale λ_s. In such cases we expect that compactified dimensions and/or universes can be excited and, therefore, that probability — in our original scattering process — is lost for good reasons: in order to recover it we would need scattering data in some extra world.[10] Whichever the case, we

[9] For instance, we expect the wave-fronts of the solution (1) to be shifted [23] when they merge each other just to make their motion consistent with the scattering process.

[10] An example of this kind is perhaps provided by the recent analysis of string-brane scattering [24] in the case that the metric potential is singular enough to allow fall into the center.

are inclined to think that, for $b < b_c$, solutions with $\rho(0) \neq 0$, depending on the string dynamics, can play an important role in explaining the unitarity loss of the present model.

6 Discussion

The first and most detailed outcome of this paper is that the reduced-action model for the transplanckian S-matrix, valid in the regime $R, b \gg \lambda_s$, shows a unitarity suppression in the region $b < b_c$, that we think corresponds to classical gravitational collapse. This feature, originally found at semiclassical level in [7], is here confirmed at more general quantum level in sec. 4 and by a perturbative method around the critical region in sec. 3. We find in particular that the unitarity defect around the critical point is non analytic with a fractional critical exponent when inelastic emission is forbidden ($y = 0$), while it shows a more regular behaviour when $y > 0$.

Since such unitarity defect raises questions about the information loss in gravitational collapse, we have tried to investigate also whether the present model is complete and if not, how to complete it. Let us first note that the model has a hermitian lagrangian and is thus expected to be unitary, barring some special conditions. But a key role is played by the boundary condition $\rho(0) = 0$. On one hand, it is needed in order to provide UV-safe solutions with regular metric, so as to insure that the eikonal is a perturbative series in R^2/b^2 for $b > b_c$, independently of the string scale λ_s. On the other hand, that condition is the main cause of the unitarity defect for $b < b_c$ because, in order to reach $\rho = 0$ at $r^2 = 0$, the amplitude has to cross a potential barrier in ρ-space by a tunnel effect, and is thereby suppressed.

For the above reasons, here we suggest to consider the solutions with $\rho(0) \neq 0$ as the alternative path that the probability flow may take. Such solutions yield a singular metric and a divergent action and probably correspond to classically trapped fields for $b < b_c$. In the toy model of sec. 5 we let $\rho(0)$ fluctuate with a weight provided by the reduced action itself, and we find that $\rho(0) \neq 0$ is violently suppressed at elastic and quasi-elastic level, thus justifying the condition $\rho(0) = 0$ in those cases. Nevertheless, the model is formally unitary and acquires a total probability of order unity in association with a large number of hard gravitons, a process which is, however, forbidden by energy conservation.

The paradoxical features above illustrate the problem we have in looking at the $\rho(0) \neq 0$ solutions: the present model suggests that unitarity might be recovered at inelastic level, but is by itself inadequate. Therefore, we have to rely on the string dynamics and related vertices [10] in order to describe the short-distance behaviour of the solutions and their contributions to unitarity. A key point is to determine the important scales of the evolution from R to λ_s, and the kind of inelastic channels which are involved, which may include states propagating in extra dimensions or universes. In sec. 5 we have listed several string effects, some of which at scale λ_s and some at scales of order R, but only a careful analysis can tell us where the lost probability goes.

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A Matrix element to continuum states

In this appendix we explicitly perform the computation of the matrix element to continuum states of eq. (104). Starting from the integral representation (102) for ϕ_E and $\psi_c = \phi_0$ we have

$$\int_{-\infty}^{+\infty} \phi_E^* \, \rho \, \psi_c \, \mathrm{d}\rho = N(\alpha) N^*\big(\frac{\alpha}{t_0}\big) \int_{-\infty}^{+\infty} \mathrm{d}t \, \mathrm{d}p \, \mathrm{d}\rho \\ \times \rho \, \mathrm{e}^{\mathrm{i}2\alpha\rho(t - t_0 p)} \left(\frac{t-1}{t+1}\right)^{\mathrm{i}\alpha} \left(\frac{p-1}{p+1}\right)^{-\mathrm{i}\frac{\alpha}{t_0}} \frac{\mathrm{sign}(t-1)\mathrm{sign}(p-1)}{(t^2-1)(p^2-1)} \,. \tag{113}$$

The integration in ρ can be performed by introducing $\alpha_0 \equiv \frac{\alpha}{t_0}$ and rewriting the integrand as

$$\frac{\mathrm{i}}{2\alpha p} \partial_{t_0} \mathrm{e}^{\mathrm{i}2\alpha\rho(t - t_0 p)} \left(\frac{t-1}{t+1}\right)^{\mathrm{i}\alpha} \left(\frac{p-1}{p+1}\right)^{-\mathrm{i}\alpha_0} \frac{\mathrm{sign}(t-1)\mathrm{sign}(p-1)}{(t^2-1)(p^2-1)} \,. \tag{114}$$

It produces the delta function $\delta\big(2\alpha(t - t_0 p)\big)$ which is used to perform the integration in t. Apart from the N normalization factors, the matrix element is now

$$\frac{\mathrm{i}\pi}{2\alpha^2} \partial_{t_0} \int \left(\frac{t_0 p-1}{t_0 p+1}\right)^{\mathrm{i}\alpha} \left(\frac{p-1}{p+1}\right)^{-\mathrm{i}\alpha_0} \frac{\mathrm{sign}(t_0 p-1)\mathrm{sign}(p-1)}{(t_0^2 p^2-1)(p^2-1)} \, \frac{\mathrm{d}p}{p} \,. \tag{115}$$

By rescaling the variable $t_0 p \to p$ the four branch points are found at $p = \pm t_0, \pm 1$ and the integrand gets a factor t_0^2 that we conveniently rewrite as $[t_0^2 - p^2] + p^2$, obtaining

$$(115) = -\frac{\mathrm{i}\pi}{2\alpha^2} \partial_{t_0} \int \left(\frac{p-1}{p+1}\right)^{\mathrm{i}\alpha} \left(\frac{p-t_0}{p+t_0}\right)^{-\mathrm{i}\alpha_0} \frac{\mathrm{sign}(p-1)\mathrm{sign}(p-t_0)}{p^2-1} \, \frac{\mathrm{d}p}{p} \\ + \frac{\mathrm{i}\pi}{2\alpha^2} \partial_{t_0} \int \left(\frac{p-1}{p+1}\right)^{\mathrm{i}\alpha} \left(\frac{p-t_0}{p+t_0}\right)^{-\mathrm{i}\alpha_0} \frac{\mathrm{sign}(p-1)\mathrm{sign}(p-t_0)}{(p^2-1)(p^2-t_0^2)} p \, \mathrm{d}p \,. \tag{116}$$

Let us now consider the two integrals separately. If we perform the t_0 derivative in the first integral, we obtain

$$\frac{\mathrm{i}\pi}{2\alpha^2} \int \left(\frac{p-1}{p+1}\right)^{\mathrm{i}\alpha} \left(\frac{p-t_0}{p+t_0}\right)^{-\mathrm{i}\alpha_0} \frac{\mathrm{sign}(p-1)\mathrm{sign}(p-t_0)}{(p^2-1)(p^2-t_0^2)} \, \mathrm{d}p \tag{117}$$

which vanishes, being proportional to the scalar product $\langle \phi_E | \phi_0 \rangle$ of two eigenfunctions with different energy. Only the second integral remains, and it can be simplified by noting that the factor $p/(p^2 - t_0^2)$ can be obtained by a further derivative with respect to t_0. In practice,

$$(115) = \frac{\pi}{4\alpha^2 \alpha_0} \partial_{t_0}^2 I(t_0) \\ I(t_0) \equiv \int \left(\frac{p-1}{p+1}\right)^{\mathrm{i}\alpha} \left(\frac{p-t_0}{p+t_0}\right)^{-\mathrm{i}\alpha_0} \frac{\mathrm{sign}(p-1)\mathrm{sign}(p-t_0)}{p^2-1} \, \mathrm{d}p \,. \tag{118}$$

The integrand of $I(t_0)$ is built by four powers whose exponents' sum is -2. This allows us to express it in terms of the hypergeometric function, as follows. By a linear change

of integration variable $x \equiv \frac{1-t_0}{2}\frac{p+1}{p-t_0}$, we can map three of the four branch points into the "canonical" ones 0, 1, ∞:

$$I(t_0) = \frac{1}{2}\left(\frac{t_0-1}{t_0+1}\right)^{\mathrm{i}(\alpha+\alpha_0)} \int_{-\infty}^{+\infty} \mathrm{sign}(x-1)\left(\frac{x-1}{x}\right)^{\mathrm{i}\alpha} [-(1-\zeta x)]^{\mathrm{i}\alpha_0} \frac{\mathrm{d}x}{x(1-x)} \tag{119}$$

where $\zeta \equiv \frac{-4t_0}{(1-t_0)^2}$. Here the integration path is slightly shifted off the real axis so as to lie below the cut $[0,1]$ and above the cut from $-1/\zeta$ to infinity. The sign in the integrand amounts to split the integration into two pieces: one from $-\infty$ to 1 and one from $+\infty$ to 1. By rotating the latter around 1 in counterclockwise direction, both pieces ranges from $-\infty$ to 1, one below and one above the $[0,1]$ cut. In this way, we arrive at the expression

$$\begin{aligned} I(t_0) = \frac{1}{2}\left(\frac{t_0-1}{t_0+1}\right)^{\mathrm{i}(\alpha+\alpha_0)} \mathrm{e}^{\pi\alpha_0} \Bigg[& -2\cosh(\pi\alpha)\int_0^1 x^{-\mathrm{i}\alpha-1}(1-x)^{\mathrm{i}\alpha-1}(1-\zeta x)^{\mathrm{i}\alpha_0}\,\mathrm{d}x \\ & +2\int_0^\infty x^{-\mathrm{i}\alpha-1}(1+x)^{\mathrm{i}\alpha-1}(1+\zeta x)^{\mathrm{i}\alpha_0}\,\mathrm{d}x \Bigg] , \end{aligned} \tag{120}$$

where the $2\cosh(\pi\alpha)$ factor arises from evaluating the integrand above and below the $[0,1]$ cut. With the substitution $y = \frac{x}{1+x}$ in the second integral, we recognize in eq. (120) two integral representations of the hypergeometric function, yielding

$$\begin{aligned} I(t_0) = \left(\frac{t_0-1}{t_0+1}\right)^{\mathrm{i}(\alpha+\alpha_0)} \mathrm{e}^{\pi\alpha_0} \Big[& \pi\alpha_0\zeta\coth(\pi\alpha)F(1-\mathrm{i}\alpha, 1-\mathrm{i}\alpha_0; 2; \zeta) \\ & + \frac{\Gamma(-\mathrm{i}\alpha)\Gamma(1-\mathrm{i}\alpha_0)}{\Gamma\big(1-\mathrm{i}(\alpha+\alpha_0)\big)} F\big(-\mathrm{i}\alpha, -\mathrm{i}\alpha_0; 1-\mathrm{i}(\alpha+\alpha_0); 1-\zeta\big)\Big] \end{aligned} \tag{121}$$

as in eq. (105).

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